Math in Common:
Strategies for Implementation

Volume 1

MATH in common

Cultivating Excellence

Tulare County
Office of Education
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Dinuba Unified School District

Dinuba Unified School District
Math in Common: Strategies for Implementation
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Dinuba Unified School District
Math in Common: Strategies for Implementation

DUSD Mathematics Vision Statement:

Through high quality mathematics instruction and assessment, DUSD students will have the mathematics content knowledge, conceptual understanding, and problem solving ability to succeed in college and career.

Mathematics Instruction	Mathematics Assessment
• Rigorous tasks with age appropriate complexity of reasoning	• Use of Formative Assessment Processes within daily lessons in order to make immediate adjustments to instruction and learning
• Strong conceptual understanding	• Use of rigorous, standards aligned common end of unit assessments to inform instruction and learning
• Alignment to grade level content standards and the Standards for Mathematical Practice	• Use of rigorous standards-aligned, summative benchmarks three times a year to analyze achievement trends

The PLC Teaching-Assessing-Learning Cycle

1. *Grade level or department teams agree on the learning targets for the unit and design/agree upon the common unit and common assessment instrument.*
2. *Teachers implement the unit using formative assessment processes.*
3. *Students take action on in-class formative assessment feedback.*
4. *Students use formative assessment instruments for motivation, reflection and action.*
5. *Grade level or department teams use ongoing assessment feedback to improve instruction.*
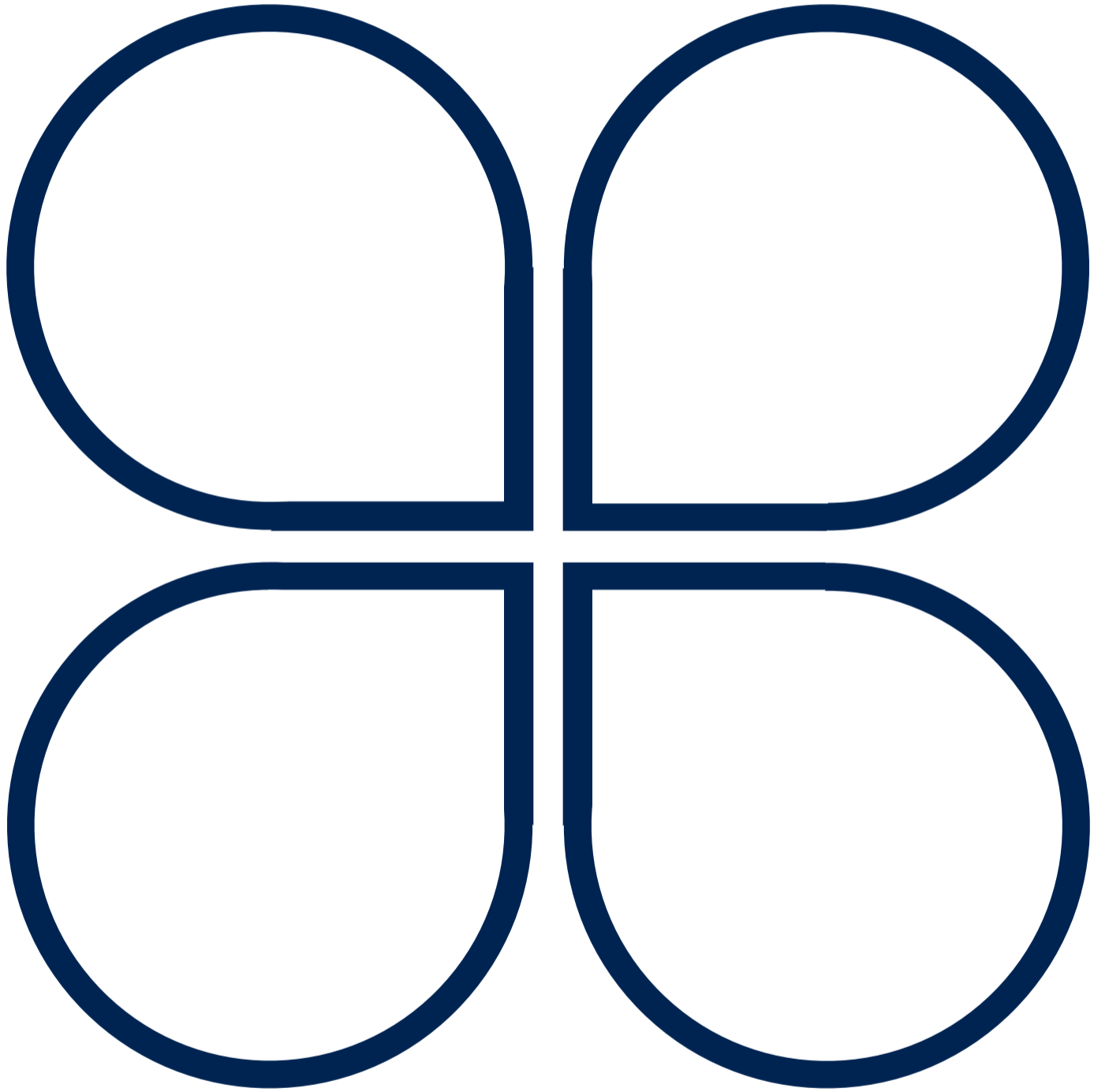
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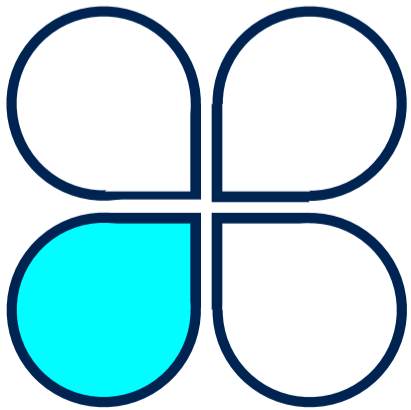
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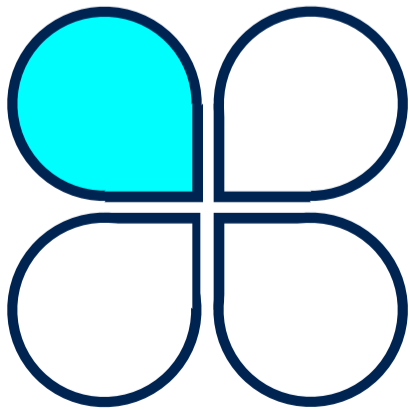
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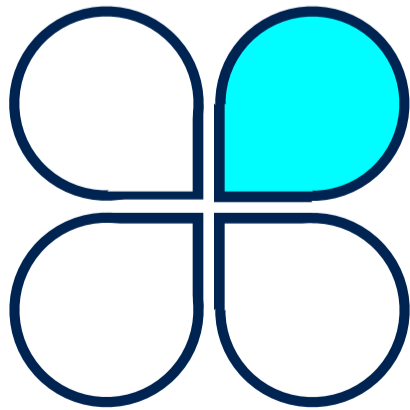
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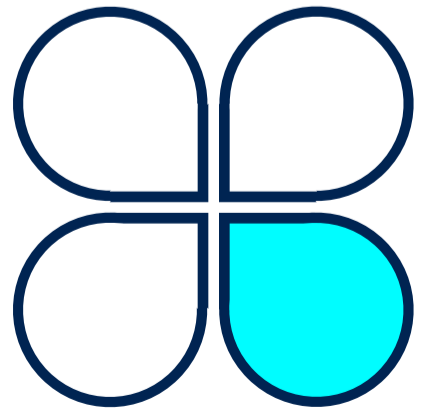
Act on Evidence

Instructional Strategies for Mathematics

The purpose of this chapter is not to prescribe the usage of any particular instructional strategy, but to enhance teachers' repertoire. Teachers have a wide choice of instructional strategies for any given instructional goal, and effective teachers look for a fit between the material to be taught and strategies to teach it. (See the grade-level and course-level chapters for more specific examples.) Ultimately, teachers and administrators must decide which instructional strategies are most effective in addressing the unique needs of individual students.

In a standards-based curriculum, effective lessons, units, or modules are carefully developed and are designed to engage all members of the class in learning activities focused on the eventual student mastery of specific standards. Such lessons, typically last at least 50 to 60 minutes daily (excluding homework). Central to the CA CCSSM and this framework is the goal that all students should be college and career ready by mastering the standards. Lessons need to be designed so that students are regularly being exposed to new information while building conceptual understanding, practicing skills, and reinforcing their mastery of previously introduced information. The teaching of mathematics must be carefully sequenced and organized to ensure that all standards are taught at some point and that prerequisite skills form the foundation for more advanced learning. However, it should not proceed in a strictly linear order, requiring students to master each standard completely before being introduced to another. Practice leading toward mastery can be embedded in new and challenging problems that promote conceptual understanding and fluency in mathematics.

Thus, teachers are presented with the following task: how to effectively deliver CA CCSSM aligned instruction that pays attention to these Key Instructional Shifts, the Standards for Mathematical Practice, and the Critical Areas of Instruction at each grade level. In this section, several instructional models are described in generality. Each has particular strengths with regard to the aforementioned instructional features. Although the classroom teacher is ultimately responsible for delivering instruction, research on how students learn in classroom settings can provide useful information to both teachers and developers of instructional resources.

The Mathematics Framework was adopted by the California State Board of Education on November 6, 2013. The Mathematics Framework has not been edited for publication.

Based upon the diversity of students found in California classrooms and the new demands of the CA CCSSM, a combination of instructional models and strategies will need to be considered to optimize student learning. Cooper (2006) lists four overarching principles of instructional design for students to achieve learning with understanding:

1. "Instruction is organized around the solution of meaningful problems.
2. Instruction provides scaffolds for achieving meaningful learning.
3. Instruction provides opportunities for ongoing assessment, practice with feedback, revision, and reflection
4. The social arrangements of instruction promote collaboration, distributed expertise, and independent learning." (Cooper 2006, 190)

Mercer and Mercer (2005) suggest that instructional models can be placed along a continuum of choices that range from explicit to implicit instruction:

Explicit Instruction	Interactive Instruction	Implicit Instruction
Teacher serves as the provider of knowledge	Instruction includes both explicit and implicit methods	Teacher facilitates student learning by creating situations where students discover new knowledge and construct own meanings
Much direct teacher assistance	Instruction includes both explicit and implicit methods	Non-direct teacher assistance
Teacher regulation of learning	Shared regulation of learning	Student regulation of learning
Directed discovery	Guided discovery	Self-discovery
Direct instruction	Strategic instruction	Self-regulated instruction
Task analysis	Balance between part-to-whole and whole-to-part	Unit approach
Behavioral	Cognitive/metacognitive	Holistic

They further suggest that the type of instructional models that will be utilized during a lesson will depend upon the learning needs of students in addition to the mathematical content that is being presented. For example, explicit instruction models may support practice to mastery, the teaching of skills, and the development of skill and procedural knowledge. On the other hand, implicit models link information to students' background knowledge, developing conceptual understanding and problem solving abilities.

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General Instructional Models

5E Model (Interactive)

Carr and his team (2009) link the 5E Model to three stages of mathematics instruction (introduce, investigate, and summarize). As its name implies, this model is based on a recursive cycle of five cognitive stages in inquiry-based learning: (a) engage, (b) explore, (c) explain, (d) elaborate, and (e) evaluate. The role of the teacher in this model is multifaceted. As a facilitator, the teacher nurtures creative thinking, problem solving, interaction, communication, and discovery. As a model, the teacher initiates thinking processes, inspires positive attitudes toward learning, motivates, and demonstrates skill-building techniques. Finally, as a guide, the teacher helps to bridge language gaps and foster individuality, collaboration, and personal growth. The teacher flows in and out of these various roles within each lesson, both as planned and as opportunities arise.

The Three-Phase Model (Explicit)

This model represents a highly structured and sequential strategy utilized in direct instruction. It has proven to be effective for teaching information and basic skills during whole class instruction. In the first phase the teacher introduces, demonstrates, or explains the new concept or strategy, asks questions, and checks for understanding. The second phase is an intermediate step designed to result in the independent application of the new concept or described strategy. Once the teacher is satisfied that the students have mastered the concept or strategy, then the third phase in implemented. In the relatively brief third phase students work independently and receive opportunities for closure. This phase also often serves in part as an assessment of the extent to which students understand what they are learning and how they use their knowledge or skills in the larger scheme of mathematics.

Singapore Math (Interactive)

Singapore math emphasizes the development of strong number sense, excellent mental-math skills, and a deep understanding of place value. It is based on Bruner's principles, a progression from concrete experience using manipulatives, to a pictorial stage, and finally to the abstract level or algorithm. This sequence gives students a solid understanding of basic mathematical concepts and relationships before they start working at the abstract level. Concepts are taught to mastery, then later revisited but not re-taught. The Singapore approach focuses on developing students who are problem solvers. There is a strong emphasis on model drawing, a visual approach to solving word problems that helps students organize information and solve problems in a step-by-step manner. Please visit http://nces.ed.gov/timss/ and http://nces.ed.gov/pubsearch/pubsinfo.asp?pubid=WWCIRMSSM09 for additional information.

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Concept Attainment Model (Interactive)

Concept attainment is an inductive model to teaching and learning that asks students to categorize ideas or objects by critical attributes. During the lesson teachers provide examples and non-examples, and then ask students to 1) develop and test hypotheses about the exemplars, and 2) analyze the thinking processes that were utilized. To illustrate, students may be asked to categorize polygons and non-polygons in a way that is based upon a pre-specified definition. Through concept attainment, the teacher is in control of the lesson by selecting, defining, and analyzing the concept beforehand, and then encouraging student participation through discussion and interaction. This strategy can be used to introduce, strengthen, or review concepts, and as formative assessment (Charles and Senter 2012).

The Cooperative Learning Model (Implicit)

Students working together to solve problems is an important component of the mathematical practice standards. Students are actively engaged in providing input and assessing their efforts in learning the content. They construct viable arguments, communicate their reasoning, and critique the reasoning of others (MP3). The role of the teacher is to guide students toward the desired learning outcomes. The cooperative learning model involves students working either in partners or in mixed-ability groups to complete specific tasks. It assists teachers in addressing the needs of the wide diversity of students that is found in many classrooms. The teacher presents the group with a problem or a task and sets up the student activities. While the students work together to complete the task, the teacher monitors progress and assists student groups when necessary (Charles and Senter 2012; Burden and Byrd 2010).

Cognitively Guided Instruction (CGI) (Implicit)

This model of instruction calls for the teacher to ask students to think about different ways to solve a problem. A variety of student-generated strategies are used to solve a particular problem such as: using plastic cubes to model the problem, counting on fingers, and using knowledge of number facts to figure out the answer. The teacher then asks the students to explain their reasoning process. They share their explanations with the class. The teacher may also ask the students to compare different strategies. Students are expected to explain and justify their strategies, and along with the teacher, take responsibility for deciding whether a strategy that is presented is viable.

This instructional model puts more responsibility on the students. Rather than simply being asked to apply a formula to several virtually identical math problems, they are challenged to use reasoning that makes sense to them in solving the problem and to find their own solutions. In addition, students are expected to publicly explain and justify their reasoning to their classmates and the teacher. Finally, teachers are required to open up their instruction to

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students' original ideas, and to guide each student according to his or her own developmental level and way of reasoning. Expecting students to solve problems using mathematical reasoning and sense-making and then explain and justify their thinking has a major impact on students' learning. For example, students who develop their own strategies to solve addition problems are likely to intuitively use the commutative and associative properties of addition in their strategies. Students using their own strategies to solve problems and justifying these strategies also contributes to a positive disposition toward learning mathematics. (http://www.wcer.wisc.edu/publications/highlights/v18n3.pdf and http://ncisla.wceruw.org/publications/reports/RR00-3.PDF).

Problem-Based Learning (Interactive)

The Standards for Mathematical Practice emphasize the importance of making sense of problems and persevering in solving them (MP.1), reasoning abstractly and quantitatively (MP.2), and solving problems that are based upon "everyday life, society, and the workplace" (MP.4). Implicit instruction models such as problem-based learning, project-based learning, and inquiry-based learning provide students with the time and support to successfully engage in mathematical inquiry by collecting data and testing hypotheses. Burden and Byrid (2010) attribute John Dewey's model of reflective thinking as the basis of this instructional model: "(a) identify and clarify a problem; (b) form hypotheses; (c) collect data (d) analyze and interpret the data to test the hypotheses; and (e) draw conclusions" (Burden and Byrid 2010, 145). These researchers suggest two different approaches can be utilized for problem-based learning. During guided inquiry, the teacher provides the data and then questions the students in an effort for them to arrive at a solution. Through unguided inquiry, students take responsibility for analyzing the data and coming to conclusions.

In problem-based learning, students work either individually or in cooperative groups to solve challenging problems with real world applications. The teacher poses the problem or question, assists when necessary, and monitors progress. Through problem-based activities, "students learn to think for themselves and show resourcefulness and creativity" (Charles and Senter 2012, 125). Martinez (2010, 149) cautions that when students engage in problem solving they must be allowed to make mistakes: "If teachers want to promote problem solving, they need to create a classroom atmosphere that recognizes errors and uncertainties as inevitable accouterments of problem solving". Through class discussion and feedback, student errors become the basis of furthering understanding and learning (Ashlock 1998). (Please see "Appendix D: Mathematical Modeling" for additional information.)

This is just a sampling of the multitude of instructional models that have been researched across the globe. Ultimately, teachers and administrators must determine what works best for their student populations. Teachers may find that a combination of several instructional approaches is appropriate in any given classroom.

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Instructional Strategies Specific to the Mathematics Classroom

As teacher progress through their career they develop a repertoire of instructional strategies. The following section discusses several instructional strategies specific to the mathematics classroom, but certainly is not an exhaustive list. Teachers are encouraged to seek out other mathematics teachers and professional learning from county offices of education, the California Mathematics Project, and other providers, as well as research the Web to continue building their repertoire.

The CA CCSSM, in particular the Standards for Mathematical Practice, expect students to demonstrate competence in making sense of problems (MP.1), constructing viable arguments (MP.3), and modeling with mathematics (MP.4). Students will be expected to communicate their understanding of mathematical concepts, receive feedback, and progress to deeper understanding. Ashlock (1998, 66) concludes that when students communicate their mathematical learning through discussions and writing, they are able to "relate the everyday language of their world to math language and to math symbols." Van de Walle (2007, 86) adds that the process of writing enhances the thinking process by requiring students to collect and organize their ideas. Furthermore, as an assessment tool, student writing "provides a unique window to students' thoughts and the way a student is thinking about an idea".

Number / Math Talks (Mental Math). Parrish (2010) describes number talks as: classroom conversations around purposefully crafted computation problems that are solved mentally. The problems in a number talk are designed to elicit specific strategies that focus on number relationships and number theory. Students are given problems in either a whole-or small-group setting and are expected to mentally solve them accurately, efficiently, and flexibly. By sharing and defending their solutions and strategies, students have the opportunity to collectively reason about numbers while building connections to key conceptual ideas in mathematics. A typical classroom number talk can be conducted in five to fifteen minutes.

During a number talk, the teacher writes a problem on the board and gives students time to solve the problem mentally. Once students have found an answer, they are encouraged to continue finding efficient strategies while others are thinking. They indicate that they have found other approaches by raising another finger for each solution. This quiet form of acknowledgement allows time for students to think, while the process continues to challenge those who already have an answer. When most of the students have indicated they have a solution and strategy, the teacher calls for answers. All answers – correct and incorrect – are recorded on the board for students to consider.

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Next, the teacher asks a student to defend their answer. The student explains his/her strategy and the teacher records the students thinking on the board exactly as the student explains it. The teacher serves as the facilitator, questioner, listener, and learner. The teacher then has another student share a different strategy and records his/her thinking on the board. The teacher is not the ultimate authority, but allows the students to have a "sense of shared authority in determining whether an answer is accurate".

Questions teachers can ask:
- How did you solve this problem?
- How did you get your answer?
- How is Joe's strategy similar to or different than Leslie's strategy?

5 Practices for Orchestrating Productive Mathematics Discussions. Smith and Stein (2011) identify five practices that assist teachers in facilitating instruction that advances the mathematical understanding of the class:
- Anticipating
- Monitoring
- Selecting
- Sequencing
- Connecting

Organizing and facilitating productive mathematics discussions for the classroom take a great deal of preparation and planning. Prior to giving a task to students, the teacher should anticipate the likely responses that students will have so that they are prepared to serve as the facilitator of the lesson. Students will usually come up with a variety of strategies, but it is helpful when leading the discussion if teachers have already anticipated some of them. The teacher then poses the problem and gives the task to the students. The teacher monitors the student responses while they work individually, in pairs, or in small groups. The teacher pays attention to the different strategies that students are using. In order to conduct the "share and summarize" portion of the lesson, the teacher selects a student to present his/her mathematical work and sequences the sharing so that the various strategies are presented in a specific order, to highlight the mathematical goal of the lesson. As the teacher conducts the discussion, the teacher is intentional about asking questions to facilitate students connecting the responses to the key mathematical ideas.

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Launch, Explore, Summarize

To effectively teach CMP2 ***several instructional shifts*** must take place:

➢ The teacher ***becomes a listener,*** rather than a talker*.

➢ The teacher ***becomes the sorter of information,*** rather than the giver of information.

➢ Students learn to ***rely on reasoning and proof,*** rather than relying on telling and patterns.

The instructional model for CMP2 is called Launch – Explore – Summarize. These three pieces define a lesson and lead to mathematical understanding.

Launch

The purpose of the launch is for the **teacher** to
- introduce new concepts.
- review old concepts.
- help students understand the context of the problem.
- issue a mathematical challenge to the students.

Suggestions for effective launches:
- Tell a story to set up the problem.
- Relate the problem to the students' lives and their activities.
- Create challenges for the students.
- Revisit ideas from previous math experiences.
- Vary the type of launch from day to day.
- Make expectations clear to the students.
- Create a clear focus.

Launches are ***not*** as effective if
- they last half the period.
- teachers or students read the book introduction to the launch <u>all</u> the time.
- the teacher models how to do the problem.
- not enough information is presented.
- the teacher questions away the problem.

Explore

The **<u>students</u>** are active during this phase and should be observed
- gathering data.
- sharing ideas.
- looking for patterns.
- making conjectures.
- developing strategies.
- creating arguments to support their reasoning and their solution.

The **<u>teacher</u>** becomes a facilitator during this phase and should be
- asking questions to encourage thought.
- asking questions to redirect.
- observing individual differences.
- providing extra challenges.

During an effective explore **<u>students</u>** should be
- choosing the tools they need.
- solving the problem.
- asking questions of each other.
- recording solutions in their notes.
- preparing a presentation.

During an effective explore **<u>teachers</u>** should be
- asking questions to redirect or extend learning.
- taking note of student strategies and solutions.
- tracking attempts, struggles, and successes.
- mentally orchestrating the summary.
- constantly making instructional decisions.

Explorations are ***not*** as effective if
- the groups are always the same people.
- there is no variation in group configuration.
- teachers do not trust their students to stay on task when working in groups.
- the teacher uses the time to tutor individual students or a few groups.
- the teacher uses the time to do desk work.
- students are not held responsible for their learning.

Summarize

During the summary, teachers and students work together to resolve the mathematics presented in the problem and lay the groundwork for future study.

As a group, **teachers and students** will
- collect, organize, and analyze data.
- observe differences and similarities.
- discuss and refine strategies.
- develop rules or generalizations.
- verify generalizations.

During an effective summary
- students present ideas.
- conversations involve the whole class.
- students debate over the correctness of answers.
- students analyze strategies and discuss similarities.
- students are encouraged to ask questions.
- The mathematics is related to previous concepts.
- upcoming mathematical ideas are foreshadowed.
- extension questions are asked.

A summary is **_not_** as effective if
- it is omitted due to time.
- everyone presents ideas.
- everything is done orally.
- every question is answered.
- the teacher tells students how it should have been done.
- students hear only correct answers.
- students speak to the teacher, not to the class.
- there is no push to think about the similarity of thoughts and strategies.

Beginning to Problem Solve with "I Notice/I Wonder"

http://mathforum.org/pow/support/activityseries/understandtheproblem.html

Basic "I Notice/I Wonder" Brainstorm

The obstacles: Students don't know how to begin solving word problems. They don't trust or make use of their own thinking. They freeze up or do any calculation that pops into their head, without thinking, "does this make sense?" They don't have ways to check their work or test their assumptions. They miss key information in the problem. They don't understand the "story" of the problem.

The solution: Create an safe environment where students focus on sharing their thoughts without any pressure to answer or solve a problem.

> Display a problem scenario or complete problem at the front of the room. If reading level is a concern, read the scenario to students or have a volunteer read it.
>
> Ask students, "What do you notice?"
>
> Pause to let as many students as possible raise their hands. Call on students and record their noticings at the front of the room.
>
> *As you record students' thoughts, thank or acknowledge each student equally. Record all student suggestions. Avoid praising, restating, clarifying, or asking questions.*
>
> Ask students, "What are you wondering?"
>
> Pause to let as many students as possible raise their hands. Call on students and record their wonderings at the front of the room.
>
> Ask students, "Is there anything up here that you are wondering about? Anything you need clarified?" If you or the students have questions about any items, ask the students who shared them to clarify them further.

Forget The Question: Access for All Students

The obstacle: Sometimes when we put a problem on the board, students notice the question and go into one of two modes:

> I don't understand, I'll never get this.
>
> I know exactly what to do, let me work as quickly as I can.

This can make it difficult to facilitate a whole-group brainstorm. The first student doesn't participate and doesn't connect to his own thinking, losing out on the power of noticing and wondering. The second student doesn't participate and narrows in too quickly on her own thinking, losing out on the opportunity to surface more interesting (and more challenging) mathematical questions and ideas.

The solution: Use the basic "I Notice/I Wonder" Brainstorm, but include only the mathematical scenario. Leave out the question, and even some key information for solving the problem. Only after all students have participated and understand the scenario thoroughly do you reveal the question. Or, ask students, "If this story were the beginning of a math problem, what could the math problem be?" Then solve a problem the *students* came up with.

Leaving off the question increases participation from struggling students because there's no right answer and no wrong noticings and wonderings. It keeps speedy students engaged in creative brainstorming rather than closed-ended problem solving. And having a question to solve that students generated increases all students' understanding of the task and their engagement.

Think/Pair/Share: Increasing Engagement and Accountability

The obstacle: Some students are shy or hesitant to participate in a brainstorming session.

The solution: Hold all students accountable by giving each a recording sheet.

The Math Forum @ Drexel: People Learning Math Together Since 1992

Students spend a minute (or more depending on their stamina) writing their noticings and wonderings on the recording sheet.

Students work with the person next to them to compare their lists and see if they can add two more things.

Each pair chooses one item to share with the whole group.

Quickly go around the room hearing each pair's items. Students should add noticings and wonderings they didn't come up with to their own sheets.

Finally ask, "Did anyone have any other noticings or wonderings they wanted to share?" and collect those.

In this fashion, each student is accountable for noticing and wondering about the problem before hearing from others, and students who are thoughtful and move slowly get a chance to organize their thoughts before sharing.

We noticed, we wondered, now what?

Noticing and wondering is a tool to help students:

Understand the story, the quantities, and the relationships in the problem.

Understand what the problem is asking and what the answer will look like.

Have some ideas to begin to solve the problem.

This means that at the end of a noticing and wondering sessions, students should be able to:

Tell the story of the problem in their own words.

Give a reasonable estimate or high and low boundaries for the answer.

Work independently on carrying out steps or generating more data toward solving the problem.

If students are not ready to do those things, we recommend any of the following activities:

PoW IQ: Describe the *Information* and *Question*. Say what you are being asked to find, and estimate an answer. Give a high and low boundary for the answer, say whether it could be negative, fractional, zero, etc. Tell the key information given in the problem that you think you will use.

Act it Out: Have a group of students act out the problem while the audience looks at their list of noticing and wondering. The audience should be prepared to share new noticings and wonderings, as well as tell if the group missed or changed any noticings.

Draw a Picture: Have each student draw a sketch that they think shows what happens in the problems. They should sketch first and then label their picture. Students can then use their sketches to say the problem in their own words to a partner or small group.

Are we done noticing and wondering yet?

Nope! Noticings and wonderings are great tools for checking your work at the end of the problem. Students don't have to ask, "Am I correct?" They can look at their noticing, wondering, and estimates to make sure they were accountable to all the information in the problem.

And noticing and wondering is a skill students can get better at. That's why it's important to **look back over your noticings and wonderings** and ask, **"Are we getting better?"** After solving a problem, ask:

Which noticings and wonderings were really important to us?

Were there noticings and wonderings we didn't really use?

How do we come up with noticings and wonderings that are mathematical? What makes them mathematical?

Did we get stuck because we'd missed something? Why did we miss it? What could we do differently next time?

After noticing and wondering several times, ask:

Are there types of noticings and wonderings that are important? That we often miss?

Are we generating more noticings and wonderings each time? Are they getting more useful?

How do we go from noticings and wonderings to solution paths?

Structuring the Standards for Mathematical Practice

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1. Make sense of problems and persevere in solving them

6. Attend to precision
</div>

2. Reason abstractly and quantitatively

3. Construct viable arguments and critique the reasoning of others

4. Model with mathematics

5. Use appropriate tools strategically

7. Look for and make use of structure.

8. Look for and express regularity in repeated reasoning.

☐ Reasoning and explaining

☐ Modeling and using tools

☐ Seeing structure and generalizing

☐ Overarching habits of mind of a productive mathematical thinker.

McCallum, Bill. 2011. Structuring the Mathematical Practices. http://commoncoretools.me/wp-content/uploads/2011/03/practices.pdf (accessed April 1, 2013).

Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

1) Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2) Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize,* to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3) Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an

argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments. **Students build proofs by induction and proofs by contradiction. CA 3.1** (for higher mathematics only).

4) Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5) Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6) Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7) Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 × 8 equals the well-remembered 7 × 5 + 7 × 3, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2 × 7 and the 9 as 2 + 7.

They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8) Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word "understand" are often especially good opportunities to connect the practices to the content. Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential "points of intersection" between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

Summary of Standards for Mathematical Practice	Questions to Develop Mathematical Thinking
1. Make sense of problems and persevere in solving them. • Interpret and make meaning of the problem to find a.starting point..Analyze what is given in order to explain to themselves the meaning of the problem. • Plan a solution pathway instead of jumping to a solution. • Monitor their progress and change the approach if necessary. • See relationships between various representations. • Relate current situations to concepts or skills previously learned and connect mathematical ideas to one another. • Continually ask themselves, "Does this make sense?" Can understand various approaches to solutions.	How would you describe the problem in your own words? How would you describe what you are trying to find? What do you notice about...? What information is given in the problem? Describe the relationship between the quantities. Describe what you have already tried. What might you change? Talk me through the steps you've used to this point. What steps in the process are you most confident about? What are some other strategies you might try? What are some other problems that are similar to this one? How might you use one of your previous problems to help you begin? How else might you organize...represent... show...?
2. Reason abstractly and quantitatively. • Make sense of quantities and their relationships. • Decontextualize (represent a situation symbolically and manipulate the symbols) and contextualize (make meaning of the symbols in a problem) quantitative relationships. • Understand the meaning of quantities and are flexible in the use of operations and their properties. • Create a logical representation of the problem. • Attends to the meaning of quantities, not just how to compute them.	What do the numbers used in the problem represent? What is the relationship of the quantities? How is _____ related to _____? What is the relationship between _____and _____? What does_____mean to you? (e.g. symbol, quantity, diagram) What properties might we use to find a solution? How did you decide in this task that you needed to use...? Could we have used another operation or property to solve this task? Why or why not?
3. Construct viable arguments and critique the reasoning of others. • Analyze problems and use stated mathematical assumptions, definitions, and established results in constructing arguments. • Justify conclusions with mathematical ideas. • Listen to the arguments of others and ask useful questions to determine if an argument makes sense. • Ask clarifying questions or suggest ideas to improve/revise the argument. • Compare two arguments and determine correct or flawed logic.	What mathematical evidence would support your solution? How can we be sure that...? / How could you prove that...? Will it still work if...? What were you considering when...? How did you decide to try that strategy? How did you test whether your approach worked? How did you decide what the problem was asking you to find? (What was unknown?) Did you try a method that did not work? Why didn't it work? Would it ever work? Why or why not? What is the same and what is different about...? How could you demonstrate a counter-example?
4. Model with mathematics. • Understand this is a way to reason quantitatively and abstractly (able to decontextualize and contextualize). • Apply the mathematics they know to solve everyday problems. • Are able to simplify a complex problem and identify important quantities to look at relationships. • Represent mathematics to describe a situation either with an equation or a diagram and interpret the results of a mathematical situation. • Reflect on whether the results make sense, possibly improving/revising the model. • Ask themselves, "How can I represent this mathematically?"	What number model could you construct to represent the problem? What are some ways to represent the quantities? What is an equation or expression that matches the diagram, number line.., chart..., table..? Where did you see one of the quantities in the task in your equation or expression? How would it help to create a diagram, graph, table...? What are some ways to visually represent...? What formula might apply in this situation?

Summary of Standards for Mathematical Practice	Questions to Develop Mathematical Thinking
5. Use appropriate tools strategically. • Use available tools recognizing the strengths and limitations of each. • Use estimation and other mathematical knowledge to detect possible errors. • Identify relevant external mathematical resources to pose and solve problems. • Use technological tools to deepen their understanding of mathematics.	What mathematical tools could we use to visualize and represent the situation? What information do you have? What do you know that is not stated in the problem? What approach are you considering trying first? What estimate did you make for the solution? In this situation would it be helpful to use...a graph..., number line..., ruler..., diagram..., calculator..., manipulative? Why was it helpful to use...? What can using a _____ show us that _____ may not? In what situations might it be more informative or helpful to use...?
6. Attend to precision. • Communicate precisely with others and try to use clear mathematical language when discussing their reasoning. • Understand the meanings of symbols used in mathematics and can label quantities appropriately. • Express numerical answers with a degree of precision appropriate for the problem context. • Calculate efficiently and accurately.	What mathematical terms apply in this situation? How did you know your solution was reasonable? Explain how you might show that your solution answers the problem. What would be a more efficient strategy? How are you showing the meaning of the quantities? What symbols or mathematical notations are important in this problem? What mathematical language...,definitions..., properties can you use to explain...? How could you test your solution to see if it answers the problem?
7. Look for and make use of structure. • Apply general mathematical rules to specific situations. • Look for the overall structure and patterns in mathematics. • See complicated things as single objects or as being composed of several objects.	What observations do you make about...? What do you notice when...? What parts of the problem might you eliminate..., simplify...? What patterns do you find in...? How do you know if something is a pattern? What ideas that we have learned before were useful in solving this problem? What are some other problems that are similar to this one? How does this relate to...? In what ways does this problem connect to other mathematical concepts?
8. Look for and express regularity in repeated reasoning. • See repeated calculations and look for generalizations and shortcuts. • See the overall process of the problem and still attend to the details. • Understand the broader application of patterns and see the structure in similar situations. • Continually evaluate the reasonableness of their intermediate results	Explain how this strategy work in other situations? Is this always true, sometimes true or never true? How would we prove that...? What do you notice about...? What is happening in this situation? What would happen if...? Is there a mathematical rule for...? What predictions or generalizations can this pattern support? What mathematical consistencies do you notice ?

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Using the Rubric:

Review each row corresponding to a mathematical practice. Use the boxes to mark the appropriate description for your task or teacher action. The task descriptors can be used primarily **as** you develop your lesson to make sure your classroom tasks help cultivate the mathematical practices. The teacher descriptors, however, can be used **during** or **after** the lesson to evaluate how the task was carried out. The column titled "proficient" describes the expected norm for task and teacher action while the column titled "exemplary" includes all features of the proficient column and more. A teacher who is exemplary is meeting criteria in **both** the proficient and exemplary columns.

PRACTICE	NEEDS IMPROVEMENT	EMERGING (teacher does thinking)	PROFICIENT (teacher mostly models)	EXEMPLARY (students take ownership)
Make sense of problems and persevere in solving them.	Task: ☐ Is strictly procedural. ☐ Does not require students to check solutions for errors. Teacher: ☐ Does not allow for wait time; asks leading questions to rush through task. ☐ Does not encourage students to individually process the tasks. ☐ Is focused solely on answers rather than processes and reasoning.	Task: ☐ Is overly scaffolded or procedurally "obvious". ☐ Requires students to check answers by plugging in numbers. Teacher: ☐ Allots too much or too little time to complete task. ☐ Encourages students to individually complete tasks, but does not ask them to evaluate the processes used. ☐ Explains the reasons behind procedural steps. ☐ Does not check errors publicly.	Task: ☐ Is cognitively demanding. ☐ Has more than one entry point. ☐ Requires a balance of procedural fluency and conceptual understanding. ☐ Requires students to check solutions for errors using one other solution path. Teacher: ☐ Allows ample time for all students to struggle with task. ☐ Expects students to evaluate processes implicitly. ☐ Models making sense of the task (given situation) and the proposed solution.	Task: ☐ Allows for multiple entry points and solution paths. ☐ Requires students to defend and justify their solution by comparing multiply solution paths. Teacher: ☐ Differentiates to keep advanced students challenged during work time. ☐ Integrates time for explicit meta-cognition. ☐ Expects students to make sense of the task and the proposed solution.

RUBRIC – IMPLEMENTING STANDARDS FOR MATHEMATICAL PRACTICE

PRACTICE	NEEDS IMPROVEMENT	EMERGING (teacher does thinking)	PROFICIENT (teacher mostly models)	EXEMPLARY (students take ownership)
Reason abstractly and quantitatively.	Task: ☐ Lacks context. ☐ Does not make use of multiple representations or solution paths. Teacher: ☐ Does not expect students to interpret representations. ☐ Expects students to memorize procedures with no connection to meaning.	Task: ☐ Is embedded in a contrived context. Teacher: ☐ Expects students to model and interpret tasks using a single representation. ☐ Explains connections between procedures and meaning.	Task: ☐ Has realistic context. ☐ Requires students to frame solutions in a context. ☐ Has solutions that can be expressed with multiple representations. Teacher: ☐ Expects students to interpret and model using multiple representations. ☐ Provides structure for students to connect algebraic procedures to contextual meaning. ☐ Links mathematical solution with a question's answer.	Task: ☐ Has relevant realistic context. Teacher: ☐ Expects students to interpret, model, and connect multiple representations. ☐ Prompts students to articulate connections between algebraic procedures and contextual meaning.
Construct viable arguments and critique the reasoning of others.	Task: ☐ Is either ambiguously stated. Teacher: ☐ Does not ask students to present arguments or solutions. ☐ Expects students to follow a given solution path without opportunities to make conjectures.	Task: ☐ Is not at the appropriate level. Teacher: ☐ Does not help students differentiate between assumptions and logical conjectures. ☐ Asks students to present arguments but not to evaluate them. ☐ Allows students to make conjectures without justification.	Task: ☐ Avoids single steps or routine algorithms. Teacher: ☐ Identifies students' assumptions. ☐ Models evaluation of student arguments. ☐ Asks students to explain their conjectures.	Teacher: ☐ Helps students differentiate between assumptions and logical conjectures. ☐ Prompts students to evaluate peer arguments. ☐ Expects students to formally justify the validity of their conjectures.

RUBRIC – IMPLEMENTING STANDARDS FOR MATHEMATICAL PRACTICE

PRACTICE	NEEDS IMPROVEMENT	EMERGING (teacher does thinking)	PROFICIENT (teacher mostly models)	EXEMPLARY (students take ownership)
Model with mathematics.	Task: ☐ Requires students to identify variables and to perform necessary computations. Teacher: ☐ Identifies appropriate variables and procedures for students. ☐ Does not discuss appropriateness of model.	Task: ☐ Requires students to identify variables and to compute and interpret results. Teacher: ☐ Verifies that students have identified appropriate variables and procedures. ☐ Explains the appropriateness of model.	Task: ☐ Requires students to identify variables, compute and interpret results, and report findings using a mixture of representations. ☐ Illustrates the relevance of the mathematics involved. ☐ Requires students to identify extraneous or missing information. Teacher: ☐ Asks questions to help students identify appropriate variables and procedures. ☐ Facilitates discussions in evaluating the appropriateness of model.	Task: ☐ Requires students to identify variables, compute and interpret results, report findings, and justify the reasonableness of their results and procedures within context of the task. Teacher: ☐ Expects students to justify their choice of variables and procedures. ☐ Gives students opportunity to evaluate the appropriateness of model.
Use appropriate tools strategically.	Task: ☐ Does not incorporate additional learning tools. Teacher: ☐ Does not incorporate additional learning tools.	Task: ☐ Lends itself to one learning tool. ☐ Does not involve mental computations or estimation. Teacher: ☐ Demonstrates use of appropriate learning tool.	Task: ☐ Lends itself to multiple learning tools. ☐ Gives students opportunity to develop fluency in mental computations. Teacher: ☐ Chooses appropriate learning tools for student use. ☐ Models error checking by estimation.	Task: ☐ Requires multiple learning tools (i.e., graph paper, calculator, manipulatives). ☐ Requires students to demonstrate fluency in mental computations. Teacher: ☐ Allows students to choose appropriate learning tools. ☐ Creatively finds appropriate alternatives where tools are not available.

RUBRIC – IMPLEMENTING STANDARDS FOR MATHEMATICAL PRACTICE

PRACTICE	NEEDS IMPROVEMENT	EMERGING (teacher does thinking)	PROFICIENT (teacher mostly models)	EXEMPLARY (students take ownership)
Attend to precision.	Task: ☐ Gives imprecise instructions. Teacher: ☐ Does not intervene when students are being imprecise. ☐ Does not point out instances when students fail to address the question completely or directly.	Task: ☐ Has overly detailed or wordy instructions. Teacher: ☐ Inconsistently intervenes when students are imprecise. ☐ Identifies incomplete responses but does not require student to formulate further response.	Task: ☐ Has precise instructions. Teacher: ☐ Consistently demands precision in communication and in mathematical solutions. ☐ Identifies incomplete responses and asks student to revise their response.	Task: ☐ Includes assessment criteria for communication of ideas. Teacher: ☐ Demands and models precision in communication and in mathematical solutions. ☐ Encourages students to identify when others are not addressing the question completely.
Look for and make use of structure.	Task: ☐ Requires students to automatically apply an algorithm to a task without evaluating its appropriateness. Teacher: ☐ Does not recognize students for developing efficient approaches to the task. ☐ Requires students to apply the same algorithm to a task although there may be other approaches.	Task: ☐ Requires students to analyze a task before automatically applying an algorithm. Teacher: ☐ Identifies individual students' efficient approaches, but does not expand understanding to the rest of the class. ☐ Demonstrates the same algorithm to all related tasks although there may be other more effective approaches.	Task: ☐ Requires students to analyze a task and identify more than one approach to the problem. Teacher: ☐ Facilitates all students in developing reasonable and efficient ways to accurately perform basic operations. ☐ Continuously questions students about the reasonableness of their intermediate results.	Task: ☐ Requires students to identify the most efficient solution to the task. Teacher: ☐ Prompts students to identify mathematical structure of the task in order to identify the most effective solution path. ☐ Encourages students to justify their choice of algorithm or solution path.

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RUBRIC – IMPLEMENTING STANDARDS FOR MATHEMATICAL PRACTICE

PRACTICE	NEEDS IMPROVEMENT	EMERGING (teacher does thinking)	PROFICIENT (teacher mostly models)	EXEMPLARY (students take ownership)
Look for and express regularity in repeated reasoning.	Task: ☐ Is disconnected from prior and future concepts. ☐ Has no logical progression that leads to pattern recognition. Teacher: ☐ Does not show evidence of understanding the hierarchy within concepts. ☐ Presents or examines task in isolation.	Task: ☐ Is overly repetitive or has gaps that do not allow for development of a pattern. Teacher: ☐ Hides or does not draw connections to prior or future concepts.	Task: ☐ Reviews prior knowledge and requires cumulative understanding. ☐ Lends itself to developing a pattern or structure. Teacher: ☐ Connects concept to prior and future concepts to help students develop an understanding of procedural shortcuts. ☐ Demonstrates connections between tasks.	Task: ☐ Addresses and connects to prior knowledge in a non-routine way. ☐ Requires recognition of pattern or structure to be completed. Teacher: ☐ Encourages students to connect task to prior concepts and tasks. ☐ Prompts students to generate exploratory questions based on current task. ☐ Encourages students to monitor each other's intermediate results.

ACHIEVE THE CORE

Key Shifts of the Common Core State Standards in Mathematics

1. Focus strongly where the Standards focus	**Focus:** The Standards call for a greater focus in mathematics. Rather than racing to cover topics in today's mile-wide, inch-deep curriculum, teachers use the power of the eraser and significantly narrow and deepen the way time and energy is spent in the math classroom. They focus deeply on the major work of each grade so that students can gain strong foundations: solid conceptual understanding, a high degree of procedural skill and fluency, and the ability to apply the math they know to solve problems inside and outside the math classroom.
2. Coherence: think across grades, and link to major topics within grades	**Coherence:** Thinking across grades: The Standards are designed around coherent progressions from grade to grade. Principals and teachers carefully connect the learning across grades so that students can build new understanding onto foundations built in previous years. Teachers can begin to count on deep conceptual understanding of core content and build on it. Each standard is not a new event, but an extension of previous learning.
	Linking to major topics: Instead of allowing additional or supporting topics to detract from the focus of the grade, these topics can serve the grade level focus. For example, instead of data displays as an end in themselves, they support grade-level word problems.
3. Rigor: in major topics pursue: • conceptual understanding, • procedural skill and fluency, and • application with equal intensity.	**Conceptual understanding:** The Standards call for conceptual understanding of key concepts, such as place value and ratios. Teachers support students' ability to access concepts from a number of perspectives so that students are able to see math as more than a set of mnemonics or discrete procedures.
	Procedural skill and fluency: The Standards call for speed and accuracy in calculation. Teachers structure class time and/or homework time for students to practice core functions such as single-digit multiplication so that students have access to more complex concepts and procedures.
	Application: The Standards call for students to use math flexibly for applications. Teachers provide opportunities for students to apply math in context. Teachers in content areas outside of math, particularly science, ensure that students are using math to make meaning of and access content.

achievethecore.org

K – 6 Operations Coherence

K-6 Coherence within Addition and Subtraction

	K	1st Grade	2nd Grade	3rd Grade	4th Grade	5th Grade	6th Grade
Addition/ Subtraction	K.OA.2 Solve addition and subtraction word problems, and add and subtract within 10 by *using objects or drawings to represent the problem.* K.OA.4 For any number from 1 to 9, find the number that makes 10 when added to the given number, *e.g., by using objects or drawings, and record the answer with a drawing or equation.* K.OA.5 **Fluently** add and subtract within 5.	1.OA.6 Add and subtract within 20, demonstrating **fluency** for addition and subtraction within 10. Use *strategies such as counting on; making ten; decomposing a number leading to a ten; using the relationship between addition and subtraction; and creating equivalent but easier or known sums.* 1.NBT.4 Add and subtract within 100 using *concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.*	2.OA.2 **Fluently** add and subtract within 20 using *mental strategies. (Mental strategies listed in 1.OA.6).* By the end of Grade 2, know from memory all sums of two one-digit numbers. 2.NBT.5 **Fluently** add and subtract within 100 using *strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.* 2.NBT.7 Add and subtract within 1000, using concrete *models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.*	3.NBT.2 **Fluently** add and subtract within 1000 using *strategies* and algorithms *based on place value, properties of operations, and/or the relationship between addition and subtraction.*	4.NBT.4 **Fluently** add and subtract multi-digit whole numbers using the <u>standard algorithm.</u>	5.NBT.7 Add, subtract, multiply, and divide decimals to hundredths, using *concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.*	6.NS.3 **Fluently** add, subtract, multiply, and divide multi-digit decimals using the <u>standard algorithm</u> for each operation.

K – 6 Operations Coherence

3-6 Coherence with Multiplication

K – 1st Grade	2nd Grade	3rd Grade	4th Grade	5th Grade	6th Grade
Foundations for Multiplication: Use of • Ten Frames, • Rows • Columns • Subitizing • 100's Chart • Skip counting by 10 • Open number line	Foundations for Multiplication: 2.OA.3 Determine whether a group of objects (up to 20) has an odd or even number of members, e.g., *by pairing objects or counting them by 2s*; write an equation to express an even number as a sum of two equal addends. 2.OA.4 Use addition to find the total number of objects arranged in rectangular arrays with up to 5 rows and up to 5 columns . . . (arrays, area model)	3.OA.3 Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem. 3.OA.7 **Fluently** multiply and divide within 100, using strategies such as the relationship between multiplication and division or properties of operations. By the end of Grade 3, know all products of two one-digit numbers.	4.NBT.5 Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.	5.NBT.5 **Fluently** multiply multi-digit whole numbers using the standard algorithm. 5.NBT.7 Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.	6.NS.B.3 **Fluently** add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.

Multiplication

K – 6 Operations Coherence

3-6 Coherence with Division

	3rd Grade	4th Grade	5th Grade	6th Grade
Division	3.OA.3 Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem. 3.OA.7 **Fluently** multiply and divide within 100, using strategies such as the relationship between multiplication and division or properties of operations. By the end of Grade 3, know all products of two one-digit numbers.	4.NBT.6 Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.	5.NBT.6 Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models. 5.NBT.7 Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.	6.NS.2 **Fluently** divide multi-digit numbers using the standard algorithm. 6.NS.B.3 **Fluently** add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.

Fluency

What is fluency?

Fluency means that students are able to complete grade level appropriate computations and procedures specified within the standards for fluency relatively fast and accurate. The CCSSM, p.6 states, "procedural fluency is skill in carrying out procedures flexibly, accurately, efficiently, and appropriately". Students should be able to approach a problem, select an appropriate strategy, and efficiently complete the calculation. While students may forget memorized facts, if they have fluency, they will be able to think flexibly about problems and try another method or approach. Students may decide to approach a problem in another way if their first method does not work and they are still demonstrating fluency. Fluency should accompany mathematical understanding, not be separate from it, so students should develop deep mathematical understanding as they are developing fluency.

How do we develop it?

Fluency can be developed through a variety of different approaches. Focused Number Talks and other daily routines that develop student understanding of mathematics and number operations can build student ideas and strategies, increasing their mathematical flexibility. Conversations about methods and strategies used help students to consider the approaches of others and make mathematical thinking visible in the classroom. Fluency building games encourage students to focus on practicing their skills while having fun.

When do we develop it?

Fluency develops through on-going experiences that provide students opportunities to think about and practice their skills. Students should engage in fluency building activities daily through routines for about 10 – 15 minutes. Concepts and strategies that increase fluency may also be taught through focused instructional lessons.

How do we assess it?

Fluency should be periodically assessed through computation assessments that assess not only if students are getting the answers right, but also how they are thinking about and approaching the problems. Fluency assessments should not be timed.

Required Fluencies in K-6

Grade	Standard	Required Fluency
K	K.OA.5	Add/subtract within 5
1	1.OA.6	Add/subtract within 10
2	2.OA.2	Single-digit sums and differences (sums by memory by end of grade)
	2.NBT.5	Add/subtract within 100
3	3.OA.7	Single-digit products and quotients (products by memory by end of grade)
	3.NBT.2	Add/subtract within 1000
4	4.NBT.4	Add/subtract within 1,000,000
5	5.NBT.5	Multi-digit multiplication
6	6.NS.2,3	Multi-digit division, Multi-digit decimal operations

Source: Achieve the Core, achievethecore.org

CCSSM Standards for Fluency

K
K.OA.5 Add/subtract within 5

1st
1.OA.6 Add/subtract within 10

2nd
2.OA.2 Single-digit sums and differences (sums by memory by end of grade)
2.NBT.5 Add/subtract within 100

3rd
3.NBT.2 Add/subtract within 1000

4th
4.NBT.4 Add/subtract within 1,000,000

3rd
3.OA.7 Single-digit products and quotients (products by memory by end of grade)

5th
5.NBT.5 Multi-digit multiplication

6th
6.NS.2, 3 Multi-digit division Multi-digit decimal operations

Strategies for Supporting ELLs in Mathematics

Strategies That Make Content Accessible	Strategies That Support Communication	Strategies that Provide Opportunities for Communication
• Activate prior knowledge.	• Create vocabulary banks.	• Facilitate whole-class discussions.
• Make manipulative materials available.	• Use sentence frames.	• Allow for small-group discussions.
• Connect symbols with words.	• Ask questions that elicit explanations.	• Utilize partner talk.
• Provide visuals.	• Design questions and prompts for different proficiency levels.	• Ask for choral responses from students.
• Pose problems in familiar contexts.	• Use prompts to support student responses.	
• Elicit nonverbal responses (e.g., thumbs-up or thumbs-down).	• Foster a positive learning community and a safe atmosphere.	
• Demonstrate and model.	• Practice wait time.	
• Modify teacher talk and draw attention to key concepts.	• Consider language and math skills, as well as social factors, when grouping students.	
• Recast/rephrase mathematical ideas and terms.	• Rephrase strategies and ideas.	
• Use native language as a resource.		

Mathematics Vocabulary by Grade Level and Domain

Below are vocabulary terms for each grade level and domain within the CCSSM. This list can serve as a guide as to words that teachers can model and use in the classroom. As students gain a firmer grasp of the concepts, they will begin internalizing and using the terms as they communicate about mathematics. This will happen for different students at different times, but it is our hope that through consistent use of appropriate mathematical terms in the classroom, we can support students understanding of the concept and development of language.

Grade	Domain	Vocabulary
TK	Counting and Cardinality	Counting, numbers, counting on, counters, objects, number names 0 – 10, subitize, order, objects, How many?, circle, line, scattered configuration, compare, is greater than, is less than, is the same as (is equal to), matching strategies, counting strategies
	OA	Compose, decompose, add, addition, plus sign, subtract, subtraction, minus, is the same as (is equal to), equal sign, total, draw, explain
	NBT	Teen numbers, number words 11 – 19, ten ones and some more ones
	Measurement and Data	Compare, attributes, length, weight, height, more of _____, less of _____, describe, sort objects, color, shape, size, groups, categories, count
	Geometry	Squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, spheres, position, above, below, beside, in front of, behind, next to, name shapes, identify, flat, two – dimensional, solid, three – dimensional
	Tools	Five frames, ten frames, rows, columns, above, below, dot cards, tape diagrams, number lines, rekenreks, counters, objects for counting and sorting, calendar cards, pattern blocks, tangrams

K	Counting and Cardinality	Counting, numbers, counting on, counters, objects, number names 0 – 10, count sequence to 100, count by 1s, count by 10s, count forward, count on, represent, one more, next number, subitize, number order, objects, How many?, circle, line, scattered configuration, compare amounts of objects, compare numbers, is greater than, is less than, is the same as (is equal to), matching strategies, counting strategies
	Operations and Algebraic Thinking	Compose, decompose, add, addition, plus sign, subtract, subtraction, minus, is the same as (is equal to), equal sign, total, draw, explain, represent, equation, number sentence, word problems, make a ten, pairs to ten, fluently
	NBT	Teen numbers, number words 11 – 19, ten ones and some more ones, compose, decompose, drawing, equation, ten ones and one, two, three, four, five, six, seven, eight, or nine ones
	Measurement and Data	Compare, describe, attributes, length, weight, height, more of _____, less of _____, describe, sort objects, color, shape, size, groups, categories, count, order groups of objects from least to greatest
	Geometry	Squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, spheres, position, above, below, beside, in front of, behind, next to, name shapes, identify, flat, two – dimensional, solid, three – dimensional, compare, analyze, create, compose, describe similarities/differences, sides, corners/vertices, length, model shapes, draw, environment, world
	Tools	Five frames, ten frames, rows, columns, above, below, dot cards, tape diagrams, number lines, rekenreks, counters, objects for counting and sorting, calendar cards, pattern blocks, tangrams, hundreds chart

1	Operations and Algebraic Thinking	Addition, subtraction, word problems, adding to, taking from, putting together, taking apart, comparing, unknown, objects, drawings, equations, symbol for the unknown, total, compose, decompose, add, addition, plus sign, subtract, subtraction, minus, is the same as (is equal to), equal sign, true equation, false equation, justify, total, draw, explain, unknown addend, make a ten, doubles/near doubles, count all, counting on, fluency, fact families/relationships between 3 whole numbers
	Number and Operations in Base Ten	Count to 120, count by 1s, count by 10s (see K), read numbers, write numbers, two-digit number, digits, tens, ones, 10 as a bundle of ten ones, 11 – 19 composed of 1 ten and one, two, three, four, five, six, seven, eight, or nine ones, decade numbers 10, 20, 30, 40, 50, 60, 70, 80, 90 refer to one, two, three, four, five, six, seven, eight, or nine tens, compare, introduce inequality symbols <, >, =, is less than, is greater than, is equal to, base ten blocks, ones – units, tens – strips, hundreds – mats, build numbers, counting on, adding up in chunks, decompose a number by place value to add, make a ten, friendly/landmark numbers, mentally/mental strategies, 10 more, 10 less, explain/justify thinking, subtracting by removal
	Measurement and Data	Measure, lengths, units, order, objects, length, compare lengths indirectly, iterating units – place units end to end, no gaps or overlaps, tell time, hours, half-hours, analog and digital clocks, hour hand, minute hand, data
	Geometry	Shapes, attributes, defining attributes, non-defining attributes, closed figures, build and draw shapes, specified attributes, compose/decompose shapes, flat, two – dimensional, solid, three – dimensional, compare, analyze, create, compose, composite shapes, describe similarities/differences, sides, corners/vertices, length, model shapes, draw, environment, world, rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles, cubes, right rectangular prisms, right circular cones, and right circular cylinders, partition, two or four equal shares, halves, half of, quarters, quarter of, whole, two equal shares, four equal shares, more shares creates smaller equal shares
	Tools	Ten frames, rows, columns, above, below, dot cards, tape diagrams, number lines, rekenreks, counters, objects for counting and sorting, calendar cards, pattern blocks, tangrams, base ten blocks, hundreds chart

2	Operations and Algebraic Thinking	Addition, subtraction, word problems, adding to, taking from, putting together, taking apart, comparing, unknown, objects, drawings, equations, symbol for the unknown, total, compose, decompose, add, addition, plus sign, subtract, subtraction, minus, is the same as (is equal to), equal sign, true equation, false equation, justify, total, draw, explain, unknown addend, make a ten, doubles/near doubles, count all, counting on, fluency, fact families/relationships between 3 whole numbers, two-step word problems, mental strategies, know from memory* (*from memory means that students can use mental strategies to arrive at a solution, it does not mean memorize or automaticity),, fluency, odd, even, pairing strategies, equation with equal addends, rectangular arrays, rows, columns, equation with equal addends
	Number and Operations in Base Ten	Place value, three-digit number, compare, inequality symbols <, >, =, base ten blocks, ones – units, tens – strips, hundreds – mats, build numbers, counting on, adding up in chunks, decompose a number by place value to add, make a ten, friendly/landmark numbers, mentally/mental strategies, explain/justify thinking, subtracting by removal, skip count, count within 1,000, count by 2s, 5s, 10s, 100s, expanded form, fluent, place value strategies, strategies based on place value, compose/decompose tens and hundreds as needed, estimation, reasonable estimates, mentally add or subtract 10/100
	Measurement and Data	Measure, estimate, lengths, standard units, select tools, rulers, yardsticks, meter sticks, measuring tapes, measure twice, relate the two measurements and compare them based on the size of the unit, estimate lengths, inches, feet, centimeters, meters, How much longer?, difference in length, length word problems, drawings, equations,
	Geometry	Recognize shapes, draw, specified/given attributes, angles, faces, vertices, equal faces, identify, triangles, quadrilaterals, pentagons, hexagons, cubes, partition, divide, cut, rectangles, circles, rows, columns, array, equal shares, equal parts, halves, half of, thirds, a third of, fourths, a fourth of, quarters, a quarter of, whole, two halves make a whole, three thirds make a whole, four fourths make a whole, equal parts do not have to have the same shape
	Tools	Ten frames, rows, columns, above, below, dot cards/array cards, tape diagrams, number lines, pattern blocks, base ten blocks, hundreds chart

3	Operations and Algebraic Thinking	Products, total, equal groups, quotients, share equally, word problems, number of groups, number of objects in each group, multiplication, division, arrays, area model, drawings, equations, symbol for the unknown, relate three whole numbers, properties of operations, commutative property*, associative property*, distributive property* (*at this grade level, students do not need to use the formal names of the properties, but teachers can model the appropriate use of these terms), unknown factor problem, relationship between multiplication and division, know from memory* (*from memory means that students can use mental strategies to arrive at a solution, it does not mean memorize or automaticity), fluently, two-step word problems, mental computation, estimation strategies, represent with a letter (variable) for the unknown, reasonableness, rounding, arithmetic patterns
	NBT	Place value, round numbers, round to the nearest 10 or 100, fluently, add, subtract, strategies, algorithms base on place value, properties of operations, relationship between addition and subtraction, multiply
	Number and Operations - Fractions	Fraction, unit fraction, partitioned, equal parts, numerator, denominator, number, number line, interval, equal, equivalent fractions, compare fractions, same size, same point on the number line, generate equivalent fractions, visual fraction model, fraction strips, compare using same numerators, compare using same denominators, compare, inequality symbols $<$, $>$, $=$, is less than, is greater than, is equal to, size, same whole, explain, justify
	Measurement and Data	Measurement, data, estimation, intervals of time, liquid volumes, masses of objects, tell time, write time, nearest minute, measure time intervals, addition, subtraction, number line diagram, measure, estimate, standard units, grams, kilograms, liters, add, subtract, multiply, divide, one-step word problems, same units, drawings, represent data, interpret data, scaled picture graph, scaled bar graph, solve, one- and two-step word problems, How many more?, How many less?, key, generate measurement data, lengths, rulers, halves, fourths, data, line plot, horizontal scale, units, geometric measurement, area, multiplication, addition, area as an attribute, plane figures, two-dimensional figures/shapes, side length, square, 1 unit, a unit square, one square unit, covered, without gaps or overlaps, counting unit squares, square cm, square m, square in, square ft, improvised units, relate area to multiplication and addition, tiling it, multiplying side lengths, real world problems, distributive property* (*at this grade level, students do not need to use the formal names of the properties, but teachers can model the appropriate use of these terms), perimeter, distinguish between area and perimeter, side lengths, unknown side length, same area with different perimeters, same perimeter with different areas.
	Geometry	Shapes, attributes, categories of shapes, share attributes, quadrilaterals, triangles, hierarchy of shapes, subcategories, partition shapes, area of each part as a unit fraction, equal parts have equal areas
	Tools	Arrays/array cards, rows, columns, tape diagrams, number lines, base ten blocks, fraction strips, pattern blocks

4	Operations and Algebraic Thinking	Products, total, equal groups, quotients, share equally, word problems, comparison problems, multiplicative comparisons, additive comparisons, number of groups, number of objects in each group, multiplication, division, arrays, area model, drawings, equations, symbol for the unknown, relate three whole numbers, properties of operations, commutative property*, associative property*, distributive property* (*at this grade level, students do not need to use the formal names of the properties, but teachers can model the appropriate use of these terms), unknown factor problem, relationship between multiplication and division, fluently, multi-step word problems, mental computation, estimation strategies, represent with a letter (variable) for the unknown, reasonableness, rounding, factors, multiples, factor pairs, prime, composite, arithmetic patterns, given a rule
	Number and Operations in Base Ten	Place value, patterns in the place value system, place value symmetry, round numbers, round to any place, fluently, add, subtract, strategies based on place value, standard algorithm, properties of operations, compare, inequality symbols <, >, =, is less than, is greater than, is equal to, size, same whole, explain, justify, illustrate, multiply, rectangular arrays, area models, divide, quotients, remainders, dividends, divisors
	Number and Operations - Fractions	Fraction, unit fraction, partitioned, equal parts, numerator, denominator, number, number line, interval, equal, equivalent fractions, compare fractions, same size, same point on the number line, generate equivalent fractions, visual fraction model, fraction strips, compare fractions with different numerators and different denominators, create common numerators or denominators, benchmark fractions, compare, inequality symbols <, >, =, is less than, is greater than, is equal to, size, same whole, explain, justify, decompose, add, subtract, mixed numbers, like/common denominators, multiply, whole number, word problems, limited to work with denominators 2, 3, 4, 5, 6, 8, 10, 12, 100, decimal fractions, decimal notation, convert decimal fractions to decimals, number line diagram
	Measurement and Data	Measurement, data, measurement conversions, know relative sizes of measurement units within one system, km, m, cm, kg, g, lb, oz, l, ml, hr, min, sec, express measurement in a larger unit in terms of a smaller unit, measurement equivalents, two-column table, conversion table, solve word problems, distance, intervals of time, liquid volumes, masses of objects, money, simple fractions or decimals, number lines, measurement scale, area/perimeter formulas, real world problems, line plot, fractions measurements (1/2, 1/4, 1/8), fraction addition/subtraction problems, geometric measurement, angles, measure angles, geometric shapes, rays, endpoint, circle, center, fraction, circular arc, 1/360 is a one-degree angle, protractor, sketch angles, angle measures as additive, decompose/compose angles, measures, equation, symbol for unknown
	Geometry	Lines, angles, classify shapes, properties of shapes including their lines and angles, line segments, rays, right angles, acute angles, obtuse angles, perpendicular lines, parallel lines, categories of shapes, special triangles, quadrilaterals, line of symmetry, line-symmetric figures
	Tools	Protractor, scale, tape diagrams, number lines, pattern blocks, base ten blocks, fraction strips

5	Operations and Algebraic Thinking	Numerical expressions, evaluate, parentheses, brackets, braces, write simple expressions, interpret expressions without solving, prime factors, prime factorization, analyze, patterns, relationships, generate numerical patterns, given rules, ordered pairs, coordinate plane	
	Number and Operations in Base Ten	Place value, patterns in the place value system, place value symmetry, 10 times the place to its right and 1/10 of the place to its left, pattern of zeros when multiplying by a power of ten, pattern of moving the decimal point when multiplying or dividing by a power of ten, exponent, base, powers of ten, read, write, compare decimals to the thousandths, inequality symbols <, >, =, is less than, is greater than, is equal to, base-ten numerals, number names, expanded form, compare based on place value, round decimals to any place, multiply using the standard algorithm, divide, quotient, dividend, divisor, properties of operations, strategies based on place value, illustrate, explain, rectangular arrays, area models, add, subtract, multiply, divide decimals to hundredths using concrete models, drawings, the relationship between addition and subtraction	
	Number and Operations - Fractions	Fraction, numerator, denominator, number line, add, subtract, unlike denominators, mixed numbers, equivalent fractions, word problems, visual fraction models, benchmark fractions, estimate, reasonableness of answers, multiply, divide, interpret fractions as the division of the numerator by the denominator, multiplication as scaling/resizing, real world problems, understand products of fractions that are greater than 1 and less than 1, divide unit fractions by whole numbers, divide whole numbers by unit fractions, create story problems, unit fractions	
	Measurement and Data	Measurement, data, measurement conversions, convert units within a given measurement system, measurement equivalents, two-column table, conversion table, solve multi-step word problems, line plot, fractions measurements (1/2, 1/4, 1/8), word problems from the line plot, geometric measurement, volume, relate volume to multiplication and addition, volume as an attribute of solid figures, three-dimensional figures, cube, unit cube, one cubic unit, solid figures can be packed without gaps or overlaps using unit cubes, rectangular prism, cubic cm, cubic in, cubic ft, improvised units, volume formulas, V = l x w x h, V = B x h	
	Geometry	Coordinate plane, coordinates, ordered pairs, perpendicular line called axes, origin, x-axis, x-coordinate, y-axis, y-coordinate, first quadrant, interpret coordinate pairs in terms of the context of the situation, classify, two-dimensional figures, properties, subcategories, categories of shapes, special triangles, quadrilaterals, hierarchy of shapes	
	Tools	Rectangular prisms, cubes, unit cubes, tape diagrams, number lines, pattern blocks, base ten blocks, fraction strips, graph paper, coordinate plane	

6	Ratios and Proportional Relationships	Ratio, ratio/rate language, unit rate, ratio reasoning, real-world problems, equivalent ratios, tables of equivalent ratios, pricing, constant speed, equivalent ratios, tape diagrams, double number line diagrams, equations, compare ratios, percent, rate per 100, convert measurement units
	The Number System	Division, divide fraction by fractions, visual fraction models, equations, relationship between multiplication and division, create a story context (problem) for a given expression, divide using the standard algorithm, add, subtract, multiply, divide using the standard algorithm, greatest common factor, distributive property, common factor, rational numbers, integers, positive/negative numbers, opposite directions/values, above/below sea level, credits/debits, etc., number line diagram, opposite signs, opposite locations, opposite of a number, signs indicate quadrants in the coordinate plane, horizontal or vertical number line, ordering numbers, absolute value, inequality statements, magnitude, distinguish comparisons of absolute value from statements about order, real-world problems, coordinate plane, four quadrants, distance between points, graph points/coordinate pairs
	Expressions and Equations	Write/evaluate numerical expressions, whole number exponents, letters/variables, write expressions with variables, identify parts of expressions, sum, term, product, factor, quotient, coefficient, describe expressions, evaluate expressions with variables, formulas, order of operations, generate/create equivalent expressions, distributive property, combine like terms, solve one-variable equations and inequalities, Which value(s) make this equation/inequality true?, real-world problems, write inequalities, graph inequalities on a number line diagram, dependent/independent variables, write an equation involving and independent and dependent variable, graphs, tables, order pairs
	Geometry	Area, surface area, volume, area of right triangles, other triangles, special quadrilateral and polygons, composing rectangles or decomposing into triangles and other shapes, real-world problems, volume, rectangular prism, fractional edge lengths, relate volume to multiplication and addition, volume as an attribute of solid/three-dimensional figures, cube, unit cube, one cubic unit, solid figures can be packed without gaps or overlaps using unit cubes, rectangular prism, cubic cm, cubic in, cubic ft, improvised units, volume formulas, multiplying edge lengths, $V = l \times w \times h$, $V = B \times h$, draw polygons in the coordinate plane, coordinate, vertices, length of sides, nets for three-dimensional figures, surface area using nets
	Statistics and Probability	Statistics, statistical variability, statistical questions, anticipate variability, data collected, data distribution, center, spread, overall shape, gaps, peaks, valleys, clusters, skew, skewed left, skewed right, symmetrical, measures of center, summarizes data, display data, number line, dot plots, histograms, box plots (box and whisker plots), number of observations, attribute being investigated, method of measurement, unit of measurement, median, mean, interquartile range, range, mean absolute deviation, choose measures of center, explain/justify
	Tools	Rectangular prisms, cubes, unit cubes, tape diagrams, number lines, fraction strips, graph paper, coordinate plane, double number lines, algebra tiles, tables of equivalent ratios

Mathematics Routines

When?	Daily
How long?	10 – 15 minutes This time may begin or end your math time/period or it may be a separate time within your school day.
Why?	Daily mathematics routines strengthen students' conceptual knowledge, strategies for operations, and content knowledge over time. Teachers plan for routines strategically to deepen student understanding and clarify misconceptions by having time to explore them further. Routines offer the opportunity to pre-teach (plant seeds for concepts to be taught later), reteach, and spiral content throughout the year.
What?	Routines can be in a variety of formats and options. Teachers select and plan routines based on student needs and the current mathematical content being taught. • Number Talks* • Calendar Routines • What's My Place? What's My Value?* • Fluency Games • Choral Counting • Problem of the Day • Counting Circles • Shape Talks
What it looks like?	Routines are a safe place for students to explore mathematics, trying out new ideas and clarifying misconceptions. • Special location • Student talk drives the routine • Non-verbal cues/hand signals • Teacher/student recording makes the student thinking visible to the class • Teacher facilitation through questioning

* DUSD focus routines.

Number Talks*	An image or problem is shown to the students to think about and solve mentally. The teacher facilitates classroom conversation where students are able to explain their thinking aloud, while the teacher records the thinking to make it visible to all students in the classroom.
What's My Place? What's My Value?*	A number of the day is built, sketched and explored through a variety of structured prompts. An open prompt through "All About a Number" can also be given to ask students to think about and share everything they know about a particular number. Teachers plan purposeful prompts to support grade level math content standards and foster the development of the Math Practices.
Choral Counting	As a whole group, students chant a given counting sequence within a range of numbers that the class is working on. The teachers will ask the class to clap in front of their chests for when they count by 1s and clap over their heads when they reach each new ten (decade number). Over time, increase the range of the numbers being used and the multiples identified by clapping overhead. Discuss patterns that students notice while clapping. Choose numbers appropriate to your grade level standards (i.e. 3rd – 5th grade classes might count by 1/4s, clapping overhead on each whole number 1/4, 2/4, 3/4, 4/4 (clap overhead)). (Adapted from https://www.illustrativemathematics.org/illustrations/360)
Counting Circles	Have students stand and form a circle facing each other or stand up and count around the room. Select a counting sequence to be practiced with no more than 8-10 numbers in the sequence. Have the students start counting around the circle one by one until the last number in the sequence is reached. When the last number is reached all students clap and that student sits down. Start the counting sequence over again until another student reaches the number at the end of the sequence; everyone claps and that student sits down as well. Continue for several rounds. Number sequences should be picked that are reflective of the numbers being used within the grade level standards. (Adapted from https://www.illustrativemathematics.org/illustrations/359)
Calendar Routines	The teacher facilitates daily counting routines and math conversations during calendar time. These may include counting the days of the month, building today's date, or using the math calendar cards to focus on a particular concept for the month.
Fluency Games	Games that promote the use of the four operations encourage students to become relatively fast and accurate in a fun engaging way. Examples of fluency games are: counting collections, math bingo, snap, race to the top, race to zero, multiplication war, integer war, etc.
Problem of the Day	A word problem is posed to the class. Students engage in solving the problem using a variety of strategies. Students share their thinking and approach to the problem with partners and/or the whole class.
Shape Talks	Similar to Number Talks, a shape is shown to the class and students think about everything they know about the shape. Students share attributes and properties of the shape and the teacher records them for the class. In upper grades, students may calculate the area, perimeter, volume or surface area of given shapes after the attributes and properties have been shared.

* DUSD focus routines.

Number Talks*

When?	Daily
How long?	10 – 15 minutes (Can be alternated with other math routines) This time may begin or end your math time/period or it may be a separate time within your school day.
Why?	Number Talks develop students' ideas about numbers and number computations. Through this routine, students are able to focus on how they think about solving problems instead of focusing solely on the answer. Number Talks can also be focused on other mathematical ideas such as shapes, geometry, measurement, and word problems. Students are able to practice the Standards for Mathematical Practice as they think about, explain their thinking, and critique the reasoning of others.
What?	An image or problem is shown to the students to think about and solve mentally. The teacher facilitates classroom conversation where students are able to explain their thinking aloud, while the teacher records the thinking to make it visible to all students in the classroom. **Sample Number Talks Topics:** - Ten frames - Dot Cards - Rekenreks - Shape Talks - Addition Strategies - Subtraction Strategies - Multiplication Strategies - Division Strategies
What it looks like?	Number Talks provides students the opportunity to subitize (in the younger grades), calculate mentally, and share ideas about their thinking. - Special location - Student talk drives the routine - Non-verbal cues/hand signals - Teacher poses/shows the problem - Students think about and solve the problem - Teacher lists student answers - Teacher records student thinking - Teacher facilitation through questioning

Based on the book *Number Talks: Helping Children Build Mental Math and Computation Strategies, Grade K – 5* by Sherry Parrish,

* One of the DUSD focus routines.

Number Talks Overview
Grades TK – 6

Grade	Overview of Content and Strategies by Grade Level		
TK	• Dot images/cards • Five frames/Ten frames • Rekenreks • All About a Number • Number of the day/week • Addition		*Addition Strategies:* • Counting All • Counting On
K	• Dot images/cards • Ten frames • Rekenreks • All About a Number • Number of the day/week • Addition • Subtraction	*Addition Strategies:* • Counting All • Counting On • Doubles/Near-Doubles • Making Tens	*Subtraction Strategies:* • Adding Up • Removal • Counting Back
1	• Dot images/cards • Ten frames • Rekenreks • All About a Number • Addition • Subtraction	*Addition Strategies:* • Counting All • Counting On • Doubles/Near-Doubles • Making Tens • Landmark/Friendly Numbers • Breaking Each Number into Its Place Value • Compensation • Adding Up in Chunks	*Subtraction Strategies:* • Adding Up • Removal • Counting Back • Place Value
2	• Dot images/cards (for arrays and building concepts of multiplication) • Ten frames • Rekenreks • All About a Number • Addition • Subtraction	*Addition Strategies:* • Counting All • Counting On • Doubles/Near-Doubles • Making Tens • Landmark/Friendly Numbers • Breaking Each Number into Its Place Value • Compensation • Adding Up in Chunks	*Subtraction Strategies:* • Adding Up • Removal • Counting Back • Place Value
Tools	Dot cards, Five frames, Ten frames, Rekenreks, Number line, Open number line		

Based on the book *Number Talks: Helping Children Build Mental Math and Computation Strategies, Grade K – 5* by Sherry Parrish, * One of the DUSD focus routines.

Grade	Overview of Content and Strategies by Grade Level		
3	• Dot images/cards (for arrays and building concepts of multiplication) • All About a Number • Addition • Subtraction • Multiplication • Division	*Addition Strategies:* • Counting All • Counting On • Doubles/Near-Doubles • Making Tens • Landmark/Friendly Numbers • Breaking Each Number into Its Place Value • Compensation • Adding Up in Chunks	*Subtraction Strategies:* • Adding Up • Removal • Counting Back • Adjusting One Number to Create an Easier Problem • Keeping a Constant Difference
		Multiplication Strategies: • Repeated Addition • Skip Counting • Landmark/Friendly Numbers • Partial Products • Doubling and Halving • Breaking Factors into Smaller Factors • Area model	*Division Strategies:* • Repeated Subtraction • Sharing/Dealing Out • Partial Quotients • Multiplying Up • Proportional Reasoning
4 – 6	• Dot images/cards (for arrays and building concepts of multiplication) • All About a Number • Addition • Subtraction • Multiplication • Division	*Addition Strategies:* • Counting All • Counting On • Doubles/Near-Doubles • Making Tens • Landmark/Friendly Numbers • Breaking Each Number into Its Place Value • Compensation • Adding Up in Chunks	*Subtraction Strategies:* • Adding Up • Removal • Counting Back • Adjusting One Number to Create an Easier Problem • Keeping a Constant Difference
		Multiplication Strategies: • Repeated Addition • Skip Counting • Landmark/Friendly Numbers • Partial Products • Doubling and Halving • Breaking Factors into Smaller Factors • Area model	*Division Strategies:* • Repeated Subtraction • Sharing/Dealing Out • Partial Quotients • Multiplying Up • Proportional Reasoning
Tools	Arrays, Area Model, Number line, Open number line		

Based on the book *Number Talks: Helping Children Build Mental Math and Computation Strategies, Grade K – 5* by Sherry Parrish, * One of the DUSD focus routines.

Number Talk – Defining Features

The chart below outlines some of the core features of a Number Talk that distinguish it from any other mathematical discussion about a problem.

Facilitation Feature	Student Experience
Problems are written and read publicly, but students solve mentally (no pencil and paper or white boards)	• Students develop efficiency, accuracy and fluency with mathematical thinking using mental math. • Students move away from a reliance on standard algorithms and strict memorization, and move into sense-making and sharing their reasoning around the mathematics.
Wait time	• All students have time to reflect upon and struggle with mental math and/or come up with multiple ways of solving
Silent signals as mode of response ("I have an answer" "I have 2 strategies"…) Silent validation of who got the same answer / who agrees or disagrees with an answer	• Students are not distracted by hands in the air, or by others who have found an answer quickly and want to share immediately. • Students are motivated to come up with more than one way of solving. Emphasis is placed on the thinking process more than the answer itself. • Students interact with each other, not just with the teacher
Surface all answers up front, including mistakes	• Mistakes are treated as learning opportunities • Students agree with and/or critique the reasoning of others
Turn and Talk (optional)	• Every student has an opportunity to share her/his way of thinking about and solving the problem • Students articulate ideas with a partner before engaging in large group academic discussion
Teacher begins scribing /representing student's strategy *after* student has finished explaining and without steering student in a particular direction. Teacher confirms with the presenter that his/her thinking is properly represented.	• Multiple strategies are made public • Students see different ways to record a mental process • Scribing reflects student's actual process, and not a specific, anticipated solution path • Students feel ownership of their own strategies
Engagement /participation /comprehension questions after strategies are shared. • *Who did it exactly the same way as ___? ("raise your hand if…")* • *Can you do that? Is that legal?* • *Did everyone understand ____'s way?* • *Can someone explain ___'s strategy in your own words?* • *Who has another way of solving it?*	• Students make sense of each other's strategies • Students see multiple ways of mentally solving problems, make connections between different ways of solving problems • Students talk about their own and each other's thinking

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Number Talk Lesson Planning Template 1: Narrative

Grade Level: _____   **Unit:** _____

Core Math Idea:

Number Talk Problem:

Anticipated student methods and how to represent them:

During the Lesson

Frame for the activity: We are using a Number Talk to share different strategies for how we mentally approach a problem. Each person's role is to work on explaining their own thinking clearly, and to listen to other's explanations as well.

Maximum length of quiet time: _____

Silent signal when students are ready: _____

Process for sharing out:
-
-
-

Questions to orchestrate the class conversation about strategies:

Wrapping Up:

Number Talk Lesson Planning Template 2: Chart

Grade Level:
Core Math Idea:

Unit:

Number Talk Problem	Possible Strategies & Method of Recording	Questions to Students
Follow up Problems (series)		**Wrap Up**
Academic Language		

Number Talk Lesson Plan 1: Elementary Sample

Grade Level: 3-5   **Unit:** Multiplication and Division

Core Math Idea: Students may be hindered in this unit because they are not yet fluent with basic addition and subtraction facts. So in this Number Talk, I will focus on adding and subtracting single and double digit numbers mentally, and specifically on the idea of **doubles plus/minus one.**

Number Talk Problem(s): 15 + 16   (First in a series, to be followed by 15 +14; 20 + 21; 22 + 23; 22 + 21

Anticipated student methods and how to represent them:

Standard algorithm (stack them in your head)

$$\begin{array}{r} 15 \\ +16 \\ \hline \end{array}$$

Count on fingers: 15, 16, 17,... **31** (use open number line to represent single jumps)

Add 10 then add six
$$15+10=25$$
$$25+6=31$$

Double 15, then add one more:
$$15+15=30$$
$$30+1=31$$

Add 10 and 10, then add 5, then add 6
$$10+10=20$$
$$20+5=25$$
$$25+6=31$$

During the Lesson

Frame for the activity: We are using a Number Talk to share different strategies for how we mentally approach a problem. Each person should be ready to explain their process, and to listen to understand someone else's.

Maximum length of quiet time: 2 min

Silent signal when students are ready: Thumb up in front of your chest when you have an answer. Raise another finger for each different strategy you think of.

Process for sharing out:
- Talk to your partner about your strategy.
- Volunteers, what number did you get for your solution? (Record all responses)
- After sharing, poll the class – raise hand if you got this value

Questions to orchestrate the class conversation about strategies
- Who would like to share how they got their answer?
- I heard you say _____, did I hear correctly?
- Did anyone use a different method?
- Can someone explain _____'s strategy in their own words?
- Please raise your hand if you understand what _____ just shared.

Wrapping Up: Questions I might ask:
- Can you find two strategies that are similar? How are they the same?
  - Look at all of these strategies. Which new strategy would you want to try to use tomorrow

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Number Talk Lesson Plan 2: Elementary Sample

Grade Level: 3-5   **Unit:** Multiplication and Division

Core Math Idea: *Adding and subtracting single and double digit numbers mentally, and specifically the idea of **doubles plus/minus one**.*

Number Talk Problem	Possible Strategies & Method of Recording	Questions to Students
15 + 16	Standard algorithm (stack in your head): $\quad$ 15 $\underline{+16}$ Count on fingers: $\quad$ 15, 16, 17,... 31 *(use open number line to represent single jumps)* Add 10 then add six: $\quad$ $15 + 10 = 25$ $\quad$ $25 + 5 = 31$ Double 15, then add one more: $\quad$ $15 + 15 = 30$ $\quad$ $30 + 1 = 31$ Add 10 and 10, then add 5, then add 6 $\quad$ $10 + 10 = 20$ $\quad$ $20 + 5 = 25$ $\quad$ $25 + 6 = 31$	• Who would like to share how they got their answer? • I heard you say _____, did I hear correctly? • Did anyone use a different method? • Can someone explain _____'s strategy in their own words? • Please raise your hand if you understand what _____ just shared. **Wrap Up** Can you find two strategies that are similar? How are they the same? Look at all of these strategies. Which strategy would you want to try to use tomorrow?

Follow up Problems (series)
15 + 14 20 + 21 22 + 23 22 + 21

Academic Language
addend, sum, total, doubles, doubles plus one, doubles minus one

Number Talk Lesson Plan 1: Secondary Sample

Grade Level: *5th through 11th*   **Unit:** *Equations and Expressions*

Core Math Idea: *Modeling real world situations with expressions, equivalent expressions*

Number Talk Problem:

Given a 10 x 10 grid, what is the area of the border?
(Show students the diagram)
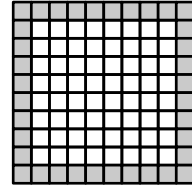
Anticipated Methods and how to represent them:

Saw four strips of 10 around the perimeter but realized that corners were counted twice.	$10 + 10 + 10 + 10 - 10 - 1 - 1 - 1$ **or** $4(10) - 4(1)$
Saw 4 center strips of 8 and added in the corners *How did you know there were 8 in a strip	$4(8) + 4(1)$ $4(10 - 1 - 1) + 4(1)$
Saw two strips of 10, one at the top, one at the bottom, and that left two strips of 8 on the right and left side. *How did you know there were 8 on the right and left?	$2(10) + 2(8)$ $2(10) + 2(10 - 2)$
Saw the strips of 10 on the right and left sides.	$2(10 - 2) + 2(10)$

During the Lesson

Frame for the activity: *"We are using a Number Talk to share different strategies for how we mentally approach a problem. Each person's role is to work on explaining their own thinking clearly, and to listen to other's explanations as well."*

Maximum length of quiet time: *3 minutes*

Silent signal when students are ready: *Fist to chest when you have an answer. Show on your fingers how many methods you can think of.*

Process for sharing out:
- Turn and Talk about your strategy
- Popcorn out, what number did you get for the area of the border? (Record all responses on board)
- Raise your hand if you got this value.

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Questions to orchestrate the class conversation about strategies: (10 minutes)

- Who would like to share how they got their solution? (LISTEN, consider how to scribe expression.)
- I heard you say _____, is that correct? (Get affirmation, then SCRIBE.)
- Please raise your hand if you understand what _____ just shared.
- Did anyone use a different method?
- Can someone explain _____'s strategy in their own words? (LISTEN, consider how to scribe expression.)
- _____, did _____ explain your method correctly? (Get affirmation, then SCRIBE.)
- Does someone have another strategy? (LISTEN, consider how to scribe expression. Student may need to approach the diagram to motion through their thinking)
- Can someone please repeat for me what _____ just described so that I can write it down?
- Might there be another method out there? (LISTEN, consider how to scribe expression. Student may need to approach the diagram to motion through their thinking)

Wrapping Up:

Questions I can ask:

- What do you notice about the expressions on the board? (Record full sentence statements)

  Are the expressions equivalent? How do we know? How can we check?
- *If a student has this expression [10 – 1 + 10 – 1 + 10 – 1 + 10 – 1)], what might that tell me about their strategy? What might a student who writes 4(10 – 1) have seen in the problem?
- *Will all of these strategies find the border for any square arrangement?

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Facilitation Guide for Whole Group Instruction in Math Class

Consider the strategies below when planning to encourage and support math talk in a lesson. Select one or two strategies for a lesson:

- **Provide think time and wait time**
    - allow students time to think quietly for a minute before asking them to respond
    - after you've asked a question, wait at least 10 seconds before calling on anyone ("keep the answer in your head")

- **Vary the modes of response: give students options for how to respond to your questions, such as:**
    - "show it on your fingers"
    - "turn to a partner and whisper the answer"
    - "keep it in your head"
    - "raise a quiet hand"
    - "tell your partner your answer, then ask: 'do you agree or disagree?'"
    - "put your thumb to your chest when you have a strategy"
    - "all together…"
    - "once you have an answer, try to think of another way to solve it"
    - response cards (students hold up prepared cards with "true" "false"; numbers; "A", "B", "C", "D"; or another appropriate answer

- **Ask students to think and talk about each other's math**
    - "who solved it exactly the same way?"
    - "raise your hand if you understand exactly how ___ solved it"
    - "what do you think ___ was thinking when s/he solved it this way?"
    - "do you agree or disagree? Why?"

- **Encourage student-to-student conversations**
    - Ask each participant to call on the next speaker ("choose someone who has not had a turn to speak yet")
    - Ask: "does anyone have a question for ____?" and allow them to call on each other to ask and answer questions
    - Remind students to make eye contact with the person they ask or call on

- **Offer sentence frames on a poster or sentence strip**
    - "I agree with _____ because…"
    - "I disagree with _____ because…"
    - "I think ____ solved it like that because…"
    - "____'s idea is interesting because…"

LCI Math

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Number Talks Implementation Profile of Practice

Emerging Implementation	Developing Implementation	Full Implementation
• Teacher selects problems that are below grade level in appropriateness.	• Teacher selects problems that are grade level appropriate. Ideas are not extended or connected to other learning.	• Teacher selects problems that are grade level appropriate, connecting and extending ideas within the Number Talk.
• Students are not using the hand signals or following the structure of the routine.	• Students are using the hand signals and follow the structure of the routine.	• Students easily use the hand signals and the structure of the routine is comfortable.
• Questions being asked are low level questions, eliciting only the answer and "steps."	• Questions being asked may be low to mid level. Students are not being asked to justify their thinking or make connections to other strategies.	• Students are highly engaged, listening thoughtfully and responding to other student's ideas.
• Strategies are not discussed in depth and the conversation may feel flat.	• Strategies are discussed, but connections are not made to other content students are learning.	• Questions being asked reflect a variety of depth of knowledge levels. These questions prompt additional responses and comments from the students.
• Students have limited strategies for solving.	• Students have multiple strategies for solving. Students are beginning to be able to identify the strategies that they are using.	• Strategies are discussed in depth and the conversation stimulates additional student ideas and questions.
		• Students have a variety of strategies for solving. They are able to identify the strategy being used and make connections between strategies.

What's My Place? What's My Value?*

When?	Daily
How long?	10 – 15 minutes (Can be alternated with other math routines) This time may begin or end your math time/period or it may be a separate time within your school day.
Why?	What's My Place? What's My Value? develops student understanding of the place value system and how operations work based on place value. Students are able to build and explore numbers, their place value, and their size through manipulating base ten blocks and placing numbers on the number line.
What?	A number of the day is built, sketched and explored through a variety of structured prompts. An open prompt through "All About a Number" can also be given to ask students to think about and share everything they know about a particular number. Teachers plan purposeful prompts to support grade level math content standards and foster the development of the Standards for Mathematical Practice.

Sample Prompts:
- Build the Number
- Sketch the Number
- Identify the place value
- Place on the number line
- Round to the nearest __
- Number of ones, tens, hundreds, etc.
- Other compositions of the number
- Operations with the number

What it looks like?	What's My Place? What's My Value? is a hands-on interactive routine that builds student understanding of place value and the properties of operations.

- Special location
- Students participate in completing the prompts whole class and in math journals or recording sheets
- Teacher and student recording
- Teacher facilitation through questioning

Based on the book *What's My Place? What's My Value?*
by Debby Head and Libby Pollett, * One of the DUSD focus routines.

What's My Place? What's My Value? Overview
Grades 1 – 6

Grade	Overview of Number Focus and Prompts by Grade Level
1	• Number focus: 1 – 100 • Building the concept of tens and ones • Building numbers using tens and ones • Building numbers with tens and ones in different ways • Sketching numbers using tens and ones • Counting forward from any number 1 – 120 • Place the number on the number line • Skip counting by 10s • Word problems within 20 • Composing tens • Compare two-digit numbers using the symbols >, =, and < • Add within 100: two-digit number with a one-digit number, or a two-digit number with a multiple of 10 • 10 more, 10 less • Subtract multiples of 10 from multiples of 10
2	• Number focus: 1 – 1,000 • Building the concepts of hundreds, tens, and ones • Building numbers using hundreds, tens, and ones • Building numbers using hundreds, tens, and ones in different ways • Focus on money by having a number of cents for the day: build, sketch, etc. • Sketching numbers using hundreds, tens, and ones • Counting forward from any number 1 – 1,000 • Place the number on the number line • Skip counting by 2s, 5s, 10s, 100s • Word problems within 100 • Composing /decomposing tens • Compare three-digit numbers using the symbols >, =, and < • Add up to four two-digit numbers • Add and subtract within 1,000 • 10 more, 10 less, 100 more, 100 less • Explain why addition and subtraction strategies work based on place value • Odd or even • Equal groups/arrays
3	• Number size: 1 – 1,000 • Build numbers using hundreds, tens, and ones • Building numbers using hundreds, tens, and ones in different ways • Sketch number using hundreds, tens, and ones • Counting forward from any number 1 – 1,000 • Skip counting by 2s, 5s, 10s, 100s • Equal groups, arrays • Place the number on the number line • Round to the nearest 10 or nearest 100 • Add and subtract within 1,000

4	• Number focuses:
	○ 1,000 – 1,0000,000
	○ Introduce decimal fractions to decimals 1/10 = .1, 1/100 = .01
	• Building the concepts of numbers up to 1,000,000
	• Building numbers up to 1,000,000
	• Sketching numbers up to 1,000,000
	• Understand the structure of the place value system
	• Round numbers to any place
	• Solve multi-step word problems using all 4 operations
	• Compare multi-digit numbers using the symbols >, =, and <
	• Understand decimal fractions with denominators of 10 and 100
	• Write equivalent fractions for decimal fractions and write them as decimals
	• Compare decimals to the hundredths using the symbols >, =, and <, justify ideas using a number line or other visual model
5	• Number focuses:
	○ 100,000 – 1,0000,000 (4th)
	○ Decimals to thousandths
	• Building the concepts of numbers to the thousandths place
	• Building numbers up to the thousandths place
	• Sketching numbers up to the thousandths place
	• Understand the structure of the place value system
	• Powers of ten – patterns of zeros when multiplying by powers of ten
	• Shifting of the decimal point – pattern of shifting the decimal point when multiplying or dividing decimals
	• Place whole numbers and decimals on the number line
	• Round decimals to any place
	• Solve multi-step word problems using all 4 operations
	• Compare decimals using the symbols >, =, and <
	• Add, subtract, multiply, and divide decimals using concrete models (i.e. base ten blocks)
6	• Number/concept focuses:
	○ Decimals to the thousandths
	○ Algebraic expressions – using the pieces as algebra tiles
	• Building the concepts of numbers to the thousandths place
	• Building numbers up to the thousandths place
	• Sketching numbers up to the thousandths place
	• Understand the structure of the place value system
	• Powers of ten – patterns of zeros when multiplying by powers of ten
	• Shifting of the decimal point – pattern of shifting the decimal point when multiplying or dividing decimals
	• Place whole numbers, decimals, and fractions on the number line
	• Round decimals to any place
	• Solve multi-step word problems using all 4 operations
	• Compare decimals using the symbols >, =, and <
	• Add, subtract, multiply, and divide using the standard algorithm and base ten blocks
	• Build algebraic expressions using algebra tiles/base ten blocks
	• Sketch and write expressions using algebra tiles/base ten blocks

What's My Place? What's My Value? – FIRST GRADE

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others
4. Model with mathematics.

5. Use appropriate tools strategically
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

California Common Core State Standards Mathematics	What's My Place? What's My Value?
1.OA.A Represent and solve problems involving addition and subtraction. **1.OA.B Add and subtract within 20.** **1.OA.C Work with addition and subtraction equations.**	**1.OA.1, 1.OA.2, 1.OA.3, 1.OA.6, 1.OA.8** • Use the units and strips to model addition and subtraction, sketch and record • Include word problems/situations for students to model using the pieces • Sketch to represent the putting together and taking apart/taking from, etc. • Record student thinking using equations • Students record their thinking using equations • Discuss the properties when applicable (i.e. 6+10 – 16 and 10+6= 16 because addition is commutative) **1.OA.4** • Include unknown-addend problems and situations • Model, sketch and record • Students make 10 in kinder and work with decomposing numbers **1.OA.5** • Build, sketch, and discuss the day's number • Build a particular number and then count on in the sequence • Identify the number that comes before/after, 10 more/10 less *(K.CC.1: Count to 100 by ones and by tens.* *2.NBT.2: Count within 1000; by 5s, 10s, and 100s.)* **1.OA.7** • For equations, build the left and right sides and discuss with students that both sides should show the same amount, demonstrate amounts that are and are not equal, discuss whether or not the equal sign can be used

What's My Place? What's My Value? - FIRST GRADE

California Common Core State Standards Mathematics	What's My Place? What's My Value?
1.NBT.A Extend the counting sequence. **1.NBT.B Understand place value.**	**1.NBT.1** • Build, sketch, and discuss the day's number • Build a particular number and then count on in the sequence • Identify the number that comes before/after, 10 more/10 less **1.NBT2** • Work with units and strips (K focused on units: 10 ones and some more ones, i.e. 16 is 10 ones and 6 more ones) • Discuss the tens and ones in the number, relate to kinder by building the number with all ones (units) • Sketch the model – can be sketch as all ones at first, help students move on to using 1 ten and some ones (i.e. 16 is 1 ten and 6 ones) **1.NBT.3** • Use the comparison symbols • Place numbers throughout the week on a WMP? WMV? Number line
1.NBT.C Use place value understanding and properties of operations to add and subtract	**1.NBT4** • Use the units and strips to model addition and subtraction, sketch and record • Include word problems/situations for students to model using the pieces • Sketch to represent the putting together and taking apart/taking from, etc. • Record student thinking using equations • Discuss the properties when applicable (i.e. 6+10 – 16 and 10+6= 16 because addition is commutative) **1.NBT.5, 1.NBT.6** • Build, sketch, and discuss the day's number • Build a particular number and then count on in the sequence • Identify the number that comes before/after, 10 more/10 less

What's My Place? What's My Value? – SECOND GRADE

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others
4. Model with mathematics.

5. Use appropriate tools strategically
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

California Common Core State Standards Mathematics	What's My Place? What's My Value?
2.OA.A Represent and solve problems involving addition and subtraction. **2.OA.B Add and subtract within 20.**	**2.OA.1, 2.OA.2** • Use the units and strips to model addition and subtraction, sketch and record • Include word problems/situations for students to model using the pieces • Sketch, discuss, and record various strategies for adding and subtracting, record these and demonstrate with pieces and a sketch • Students explain (verbally and in writing) their thinking and their sketch
2.OA.C Work with equal groups of objects to gain foundations for multiplication.	**2.OA.3, 2.OA.4** • Model, sketch, and discuss multiplication • Model a number times __ as __ groups of that number (i.e. 2 x 10 is two groups of ten objects, can be modeled by 2 ten strips) • Build models of numbers using an array • Utilize word problems/situations that can be acted out/built to show this concept
2.NBT.A Understanding Place Value	**2.NBT.1, 2.NBT.2, 2.NBT.3** • Daily practice • Identifying place value • Sketch • Expanded Form • Word Form • Model by adding pieces while skip counting.

What's My Place? What's My Value? - SECOND GRADE

2.NBT.A Understanding Place Value (continued)	**2.NBT.4** • Build and compare two three digit numbers • Compare today's and yesterday's number • Use comparison symbols • Place numbers throughout the week on WMP?WMV? Number line **2.NBT.5, 2.NBT.6, 2.NBT.7** • Use the units, strips, and mats to model addition and subtraction • Include word problems/situations for students to model using pieces • Sketch, discuss, and record various strategies for adding and subtracting, record these and demonstrate with the pieces and sketch • Students explain (verbally and in writing) their thinking and their sketch
2.NBT.7.1 Use estimation strategies to make reasonable estimates in problem solving. CA	**2.NBT.7.1** • Estimate prior to building or sketching, discuss student thinking and record ideas • Build, sketch, and discuss the day's number

What's My Place? What's My Value? – THIRD GRADE

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others
4. Model with mathematics.

5. Use appropriate tools strategically
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

California Common Core State Standards Mathematics	What's My Place? What's My Value?
3.OA.A Represent and solve problems involving multiplication and division	**3.OA.1** • Daily practice • Multiply by utilizing place value knowledge and multiplying each place by the factor • Double/triple the day's number (2/3 groups of today's number) **3.OA.2** • Divide by 2 or cut the number in half • Divisible by ___ **3.OA.3, 3.OA.4** • Model, sketch, and discuss multiplication/division • Model a number times __ as __ groups of that number (i.e. 2 x 10 is two groups of ten objects, can be modeled by 2 ten strips) • Build models of numbers using an array • Utilize word problems/situations that can be acted out/built to show this concept

What's My Place? What's My Value? - THIRD GRADE

3.NBT.A Use place value understanding and properties of operations to perform multi-digit arithmetic.

3.NBT.1
- Daily practice
- Identifying place value
- Rounding to the nearest _____
- WMP? WMV? Number line

3.NBT.2, 3.NBT.3
- Find the sum/difference of today's number and yesterday's number
- Sketch, discuss, and record various strategies for adding, subtracting, and multiplying. Demonstrate with the pieces and sketch.
- Students explain (verbally and in writing) their thinking and their sketch.

What's My Place? What's My Value? – FOURTH GRADE
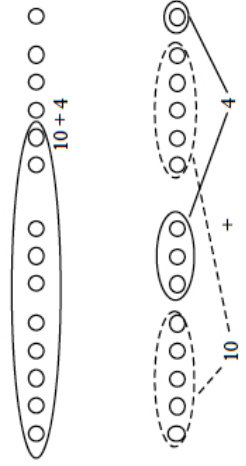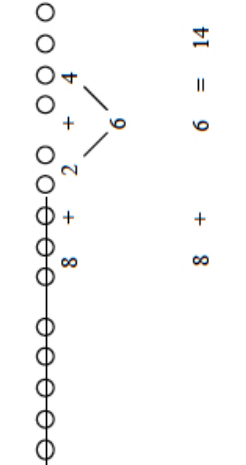
Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others
4. Model with mathematics.

5. Use appropriate tools strategically
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

California Common Core State Standards Mathematics	What's My Place? What's My Value?
4.OA.A Use the four operations with whole numbers to solve problems.	**4.OA.3** • WMP? WMV? Number line • Use the location on the number line to help round numbers • Rounding to the nearest _____ • Discuss situations where rounding would be appropriate (approximation, estimation, reasonableness, etc.) • Have students create a word problem for the number of the day. **4.OA.5** • Create a number pattern using today's number.
4.NBT.A Generalize place value understanding for multi-digit whole numbers.	**4.NBT.1, 4.NBT.2** • Daily practice • Sketch • Expanded form • Post numbers on WMP? WMV? Number line • Compare to previous day's numbers **4.NBT.3** • Use the location on the number line to help round number. • Rounding to the nearest_____
4.NBT.B Use place value understanding and properties of operations to perform multi-digit arithmetic.	**4.NBT.4, 4.NBT.5, 4.NBT.6** • Daily practice • Find the sum/difference of today's number and yesterday's number • Multiply by utilizing place value knowledge and multiplying each by place by the factor

What's My Place? What's My Value? - FOURTH GRADE

4.NBT.B Use place value understanding and properties of operations to perform multi-digit arithmetic. (continued)	• Double/triple the day's number • Divide by 2 or cut the number in half
4.NF.C Understand decimal notation for fractions and compare decimal fractions.	**4.NF.5, 4.NF.6, 4.NF.7** • Convert decimals to the tenths place or hundredths place into their fraction equivalents • Build and compare decimals to the hundredths place • WMP? WMV? Number line • Compare to previous day's numbers • Sketch • Expanded form

What's My Place? What's My Value? – FIFTH GRADE

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others
4. Model with mathematics.
5. Use appropriate tools strategically
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

California Common Core State Standards Mathematics	What's My Place? What's My Value?
5.OA.B Analyze patterns and relationships	**5.OA.3** • Create a number pattern using number of the day (i.e. input/output table: y = Today's number * X) and graph on a coordinate plane.
5.NBT.A Understand the place value system.	**5.NBT.1** • Daily practice • Sketch • Identify and explain place value • Post numbers on WMP? WMV? Number line • Compare to previous day's numbers **5.NBT.2** • Expanded notation (i.e. 347.85 = (3x100) + (4x10) + (7x1) + (8 x 1/10) + (5 x 1/100) **5.NBT.3, 5.NBT.4** • Use the number line to help students round • Compare today's and yesterday's number
5.NBT.B Perform operations with multi-digit whole numbers and with decimals to the hundredths.	**5.NBT.6, 5.NBT.7** • Daily practice • Find the sum/difference of today's number and yesterday's number • Multiply by utilizing place value knowledge and multiplying each by place by the factor • Double/triple the day's number • Divide by 2 or cut the number in half

What's My Place? What's My Value? – SIXTH GRADE

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others
4. Model with mathematics.
5. Use appropriate tools strategically
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

California Common Core State Standards Mathematics	What's My Place? What's My Value?
6.NS.B Compute fluently with multi-digit numbers and find common factors and multiples.	**6.NS.2** • Divisible by ____ **6.NS.3** • Add/subtract today and yesterday's number • Double/triple today's number • Divisible by ____
6.NS.C Apply and extend previous understandings of numbers to the system of rational numbers.	**6.NS.5, 6.NS6, 6.NS.7** • Find the opposite • Take absolute value and show its meaning as the distance from 0 • Discuss location on the number line, discuss today's number and it's opposite. • Compare and order using previous days' numbers. • Word form, Expanded form • Read, write and build numbers
6.EE.A Apply and extend previous understanding of arithmetic to algebraic expressions.	**6.EE.1, 6.EE.2** • Transition from Base 10 blocks to Algebra Tiles • Build, write, and solve the equation. • Connect story problems or real-world problems to an equation • Have students model and solve word problems and equations in the text using base 10 blocks. **6.EE.3** • Daily WMP? WMV? • Add, subtract, multiply or divide numbers. • Discuss the properties and demonstrate them with WMP? WMV? pieces.

What's My Place? What's My Value? Implementation Profile of Practice

Emerging Implementation	Developing Implementation	Full Implementation
• Teacher selects numbers and prompts that are below grade level in appropriateness.	• Teacher selects numbers and prompts that are grade level appropriate. Ideas are not extended or connected to other learning.	• Teacher selects numbers and prompts that are grade level appropriate, connecting and extending ideas within the What's My Place? What's My Value? routine.
• Students are not following the structure of the routine.	• Students are following the structure of the routine.	• Students easily use the routine independently and in small groups. Students can participate in the facilitation of the whole class routine.
• Questions being asked are low level questions – recalling and identifying.	• Questions being asked may be low to mid level. Students are not being asked to justify their thinking or make connections to other strategies.	• Students are highly engaged, thoughtfully listening and responding to other student's ideas.
• The place value work is not discussed in depth and the conversation may feel flat.	• Place value is discussed, but connections are not made to the patterns and structures within it that support a deep understanding of the operations.	• Questions being asked reflect a variety of depth of knowledge levels. These questions prompt additional responses and comments from the students.
• Students are not being stimulated by the prompts. They may be completing them without much effort and engagement is waning.	• Students are able to respond to the prompts and are beginning to make connections with place value and the operations.	• Classroom discussions stimulate additional student ideas and questions.
		• Students are able to explain their thinking as it relates to place value and the operations.

Tulare County Office of Education

Jim Vidak, County Superintendent of Schools

Supporting the Common Core State Standards for Mathematics
Strategies for Addition and Subtraction

Methods Used for Solving Single Digit Addition and Subtraction Problems
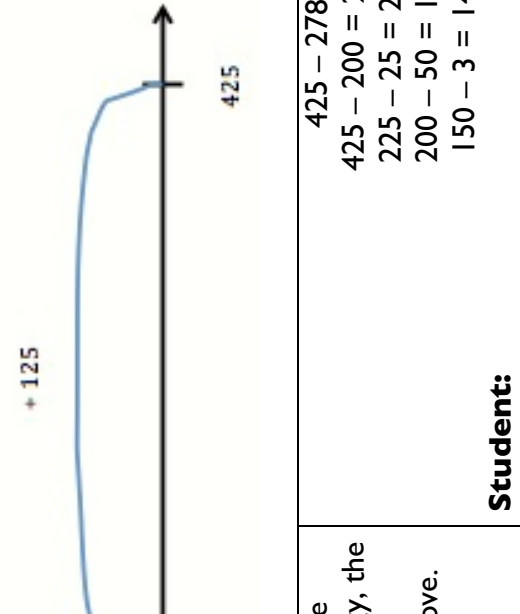
Strategy	Explanation	Examples	Supporting Standards
Direct Modeling – by Counting All or Taking Away *Level: 1	The student represents a problem situation or numerical expression using objects, a drawing or fingers. The student models combining the amounts or taking away an amount and then counts the new total.	$8 + 6 = 14$ Count All $14 - 8 = 6$ Take Away	K.CC.4, K.CC.5, K.OA.1, K.OA.2, 1.OA.1, 1.OA.2, 1.OA.5, 1.NBT.4, 2.OA.1, 2.NBT.5, 2.NBT.6, 2.NBT.7, 2.NBT.8, 2.MD.5, 3.NBT.2, 4.NF.3, 5.NF.1, 5.NF.2, 6.NS.3
Counting On *Level: 2	The student begins with of one addend and then counts on by the other addend to find the new total. Counting on can be used to find a total or an unknown addend. Students can also count back to model subtraction, but this strategy is more susceptible to errors. Counting on can happen by ones, by skip counting or by counting on in larger chunks.	Count On To solve $14 - 8$, I count on $8 + ? = 14$ I took away 8 8 to 14 is 6 so $14 - 8 = 6$ 46 +37 ____ 83 starting from 46 count on 3 tens then count on 7 ones	K.OA.5, 1.OA.1, 1.OA.2, 1.OA.4, 1.OA.5, 1.OA.6, 1.OA.8, 1.NBT.4, 2.OA.1, 2.OA.2, 2.OA.4, 2.NBT.5, 2.NBT.6, 2.NBT.7, 2.NBT.8, 3.NBT.2, 3.MD.1

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Level 3. Convert to an Easier Equivalent Problem

Decompose an addend and compose a part with another addend. These methods can be used to add or find an unknown addend (and thus to subtract). These methods implicitly use the associative property.

Make a Ten *Level: 3	The student is able to compose a ten by decomposing one of the addends in the problem situation or numerical expression. Students can easily and flexibly compose and decompose numbers leading to a ten developing fluency with their addition.	**8 + 6 = 14** Recompose: Make a Ten $10 + 4$ $8 + 6 = 8 + \underline{2} + \underline{4} = 10 + 4 + 14$	**$14 - 8 = 6$** $14 - 8$: I make a ten for $8 + ? = 14$ $8 + 2 + 4$ 6 $8 + 6 = 14$	K.OA.3, K.OA.4, 1.OA.1, 1.OA.2, 1.OA.3, 1.OA.6, 1.OA.8, 1.NBT.4, 2.OA.2, 2.NBT.5, 2.NBT.6, 3.NBT.2
Doubles/Near Doubles *Level: 3	The student is able to see the problem as a doubles fact and knows his/her doubles or can easily see a doubles fact within the problem. Knowing the double within the problem makes it easier for the student to add.	$6 + 8$ $= 6 + 6 + 2$ $= 12 + 2 = 14$		1.OA.6, 1.OA.8, 1.NBT.4, 2.OA.2, 2.OA.4, 2.NBT.5, 2.NBT.6, 3.NBT.2
Finding an Unknown Addend *Level: 3	The student thinks about what addend would be needed to get to the total in order to think about a subtraction problem using addition.	To find $14 - 8$, I can find $8 + ? = 14$. $8 + \underline{2} = 10$ $10 + \underline{4} = 14$, so $\underline{2} + \underline{4} = 6$		1.OA.6, 1.OA.8, 1.NBT.4, 1.NBT.6, 2.OA.2, 2.NBT.5, 2.NBT.6, 3.NBT.2

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Additional Strategies for Addition

Landmark or Friendly Numbers	Students replace given numbers with numbers that are close to the original problem, but are easier to compute mentally. Friendly numbers often include multiples of ten, hundred or a thousand. They may also include other numbers used often such as 25 or 75. Students may adjust one or both of the addends by adding or subtracting, then they must adjust the total they find to compensate for the friendly numbers that were created. Students do not need to use formal rounding to use the friendly number strategy, but he/she should be able to explain how they adjusted the numbers and why they chose to adjust them in this way.	$456 + 167 =$ $450 + 150 = 600$ $6 + 17 = 23$ $600 + 23 = 623$ **Student:** I changed 456 to 450 and 167 to 150 to make them easier to add. 450 and 150 is 600. Then I needed to add 6 and 17 because those are numbers that I took off when I added 450 and 150. I added 600 + 23 to get 623.
Breaking Each Number into Its Place Value	Students decompose larger numbers based on their place value understanding and then use these ideas to add hundreds with hundreds, tens with tens, and ones with ones, etc. The partial sums are added to find the grand total.	$456 + 167 = (400 + 50 + 6) + (100 + 60 + 7)$ $400 + 100 = 500$ $50 + 60 = 110$ $6 + 7 = 13$ $500 + 110 + 13 = 623$ **Student:** I broke each number into its place value (expanded form). Then I added 400 and 100 to get 500. I added 50 + 60 to get 110 and I added 6 and 7 to get 13. I added each of those (500 + 110 + 13) to get a total of 623.
Adding Up in Chunks	This strategy is similar to breaking each number into its place value, but one number is kept whole and the other is broken into easy chunks to add to it. This strategy can be shown using an open number line to illustrate the chunks that were added and why they were added that way.	$456 + 167 =$ $456 + 4 = 460$ $460 + 40 = 500$ $500 + 123 = 623$ **Student:** I started at 456. Then I added 4 to get to 460 and 40 to get 500. Since I had added 44 from the 167 so far, I needed to add 123 more. So, 500 + 123 = 623.

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| Compensation | Similar to the friendly number strategy, students will adjust the numbers to make them easier to add in the compensation strategy. The difference here is that students will remove an amount from one addend and then add that amount to the other addend to make the two numbers easier to add. Students need to think about how to adjust the addends to make the numbers easier to work with. | $456 + 167 =$

 $-33 \quad +33$

 $423 + 200 = 623$

 Student:
 I decided to add 33 to 167 to make it 200, so I needed to subtract 33 from 456. My new problem was 423 + 200, which is 623. |

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Additional Strategies for Subtraction

Adding Up	425 − 278
Students can use addition to solve subtraction problems by building up to the minuend (total). Students can utilize the idea of adding up in chunks and the tool of the open number line to record their chunks and the difference between the two numbers.	**Student:** I started at 278 and added 2 to get to 280. Then I added 20 to get to 300. Last I added 125 to get to 425. To find my answer of 147, I needed to add up all of the chunks that I added.
Removal or Counting Back	425 − 278
Students who have focused on subtraction through a take away model may lean toward this strategy. In this strategy, the students will start with the whole and then remove the subtrahend in parts that are easy for the student to remove.	$$425 - 278$$ $$425 - 200 = 225$$ $$225 - 25 = 200$$ $$200 - 50 = 150$$ $$150 - 3 = 147$$ **Student:** I subtracted in chunks. Fist I took 200 away from 425 to get 225. Then I subtracted 25 to get 200, and 50 to get 150. Since I had to subtract 278, I had to remove 3 more, so 150 − 3 = 147.

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Place Value	Students decompose larger numbers based on their place value understanding and then use these ideas to subtract hundreds with hundreds, tens with tens, and ones with ones, etc. Students may end up with a negative number or minus a given amount in a new expression to solve to find the total difference.	$425 - 278 =$ $400 + 20 + 5 - (200 + 70 + 8)$ 	400	20	5	 -	200	70	8	 	200	-50	-3	 $200 - 50 - 3 = 147$ **Student:** I subtracted each place value. Now I have $200 - 50 - 3$, so I subtracted 50 from 200 to get 150. Last, I subtracted 3 from 150 to get 147.
Keeping a Constant Difference	Students rename a given problem to create a new problem with the same difference that is easier for them to mentally subtract. Students may notice that this strategy is similar to compensation in addition, but the same amount is added to each number in order to maintain a constant difference. This can be nicely illustrated using an open number line.	$425 - 278 =$ $+22 \quad +22$ $447 - 300 = 147$ **Student:** I renamed the problem by adding 22 to each number. I added 22 to make the 2nd number into 300 so that the numbers would be easier to subtract. My new problem was $447 - 300$ which is 147.

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The Traditional/Standard Algorithm and Common Variations

Definitions from the Progressions for the Common Core State Standards in Mathematics Number and Operations in Base Ten, p. 3.		
Strategies and algorithms The Standards distinguish strategies from algorithms. For example, students use strategies for addition and subtraction in Grades K-3, but are expected to fluently add and subtract whole numbers using standard algorithms by the end of Grade 4. Use of the standard algorithms can be viewed as the culmination of a long progression of reasoning about quantities, the base-ten system, and the properties of operations.		
Computation Strategy	Purposeful manipulations that may be chosen for specific problems, may not have a fixed order, and may be aimed at converting one problem into another.

The progression distinguishes between two types of computational strategies: special strategies and general methods. For example, a special strategy for computing 398 + 17 is to decompose 17 as 2 + 15, and evaluate (398 + 2) +15. Special strategies either cannot be extended to all numbers represented in the base-ten system or require considerable modification in order or to do so. A more readily generalizable method of computing 398 + 17 is to combine like base-ten units. General methods extend to all numbers represented in the base-ten system. A general method is not necessarily efficient. For example, counting on by ones is a general method that can be easily modified for use with finite decimals. General methods based on place value, however, are more efficient and can be viewed as closely connected with standard algorithms. | K.OA.1 – 5,
1.OA.1 – 8,
1.NBT.4 – 6,
2.OA.1 – 4,
2.NBT.5 – 9,
2.MD.5, 2.MD.8,
3.OA.8 – 9,
3.NBT.2, 3.MD.1 –
2, 3.MD.7 – 8,
4.OA.3, 4.OA.6,
4.NF.3, 4.MD.2 –
3, 4.MD.7, 5.OA.3,
5.NBT.7, 5.NF.1 –
2, 5.MD.5 |
| Computation Algorithm | A set of predefined steps applicable to a class of problems that gives the correct result in every case when the steps are carried out correctly. | 3.NBT.2, 4.NBT.4,
5.NF.1 – 2, 6.NS.3 |

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Computation Algorithm for Addition	In the traditional algorithm for addition in the United States, students add from right to left adding ones and ones, tens and tens, hundreds and hundreds, etc. When a new unit is composed, a 1 is written above the next column on the left. The composing of a new unit was previously referred to as carrying and/or regrouping. This process continues until students complete their addition. **Student:** I added 6 ones and 7 ones which makes 13. I put the 3 at the bottom and 1 ten in the tens column. Then I added 1 ten, 5 tens, and 6 tens to get 12 tens. I put the 2 at the bottom and wrote the 1 in the hundreds column. $$\begin{array}{r} 1\,1 \\ 456 \\ +\,167 \\ \hline 623 \end{array}$$
Variations of the Computation Algorithm for Addition	• Adding with Recording on Separate Lines – In this variation, students add from either left to right or right to left, but the sub of each place value is recorded beneath the problem creating partial sums that will be added to find the total sum. Adding from left to right enables students to have more reasonable estimates as they are adding and coincides with the direction that they are learning to read. **Addition: Recording combined hundreds, tens, and ones on separate lines** $$\begin{array}{r} 456 \\ +\,167 \\ \hline \end{array} \quad \begin{array}{r} 456 \\ +\,167 \\ \hline 500 \end{array} \quad \begin{array}{r} 456 \\ +\,167 \\ \hline 500 \\ 110 \end{array} \quad \begin{array}{r} 456 \\ +\,167 \\ \hline 500 \\ 110 \\ 13 \\ \hline 623 \end{array}$$ *Addition proceeds from left to right, but could also have gone from right to left. There are two advantages of working left to right: Many students prefer it because they read from left to right, and working first with the largest units yields a closer approximation earlier.*

- Recording Newly Composed Units on the Same Line – In this variation, a space is left beneath the addends so that students can record the newly composed units underneath instead of above. This helps students to remember to add their newly composed units and reinforces the actual value of the addition since the digits of each sum are located in close proximity.

Addition: Recording newly composed units on the same line

```
  4 5 6          4 5 6          4 5 6          4 5 6
+ 1 6 7        + 1 6 7        + 1 6 7        + 1 6 7
                    3            1 1            1 1
                                 2 3          6 2 3
```

Add the ones, 6 + 7, and record these 13 ones with 3 in the ones place and 1 on the line under the tens column.
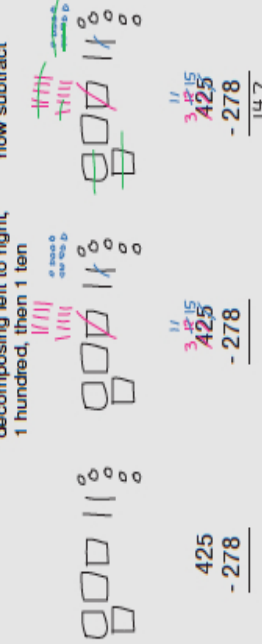
Add the tens, 5 + 6 + 1, and record these 12 tens with 2 in the tens place and 1 on the line under the hundreds column.
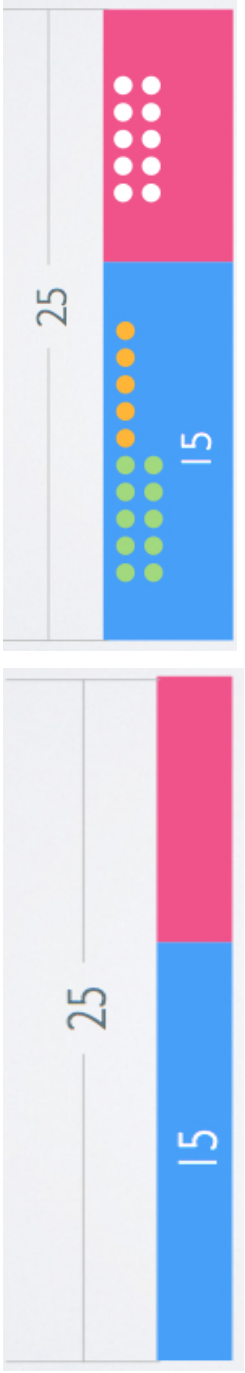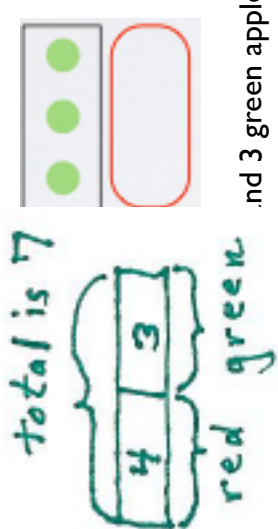
Add the hundreds, 4 + 1 + 1 and record these 6 hundreds in the hundreds column.

Digits representing newly composed units are placed below the addends. This placement has several advantages. Each two-digit partial sum (e.g., "13") is written with the digits close to each other, suggesting their origin. In "adding from the top down," usually sums of larger digits are computed first, and the easy-to-add "1" is added to that sum, freeing students from holding an altered digit in memory. The original numbers are not changed by adding numbers to the first addend; three multi-digit numbers (the addends and the total) can be seen clearly. It is easier to write teen numbers in their usual order (e.g., as 1 then 3) rather than "write the 3 and carry the 1" (write 3, then 1).

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Computation Algorithm for Subtraction	In the traditional algorithm for subtraction in the United States, students subtract from right to left subtracting ones and ones, tens and tens, hundreds and hundreds, etc. When the student needs to subtract a larger bottom number from a smaller top number, a unit from the place value position to the left is decomposed and moved to the place value being subtracted. The decomposing of a unit to aide in subtraction was previously referred to as borrowing and/or regrouping. This process continues until students complete their subtraction.
	Student:
	$$\begin{array}{r} 3\ 11\ 15 \\ 4\,2\,5 \\ -\ 2\,7\,8 \\ \hline 1\,4\,7 \end{array}$$
	I can't subtract 8 ones from 5 ones, so I cross out 2 tens and make it 1 ten. I move the other 1 ten to the right to make 15 ones. I know that 15 minus 8 is 7 so I write it at the bottom. Next, I have 1 ten minus 7 tens so I need to decompose 1 hundred. I cross out 4 hundred and make it 3 hundred. Then I change the 1 ten to 11 tens. I subtract 7 tens from 11 tens which is 4 tens and I write it down. Then I subtract 3 hundreds and 2 hundreds to get 1 hundred.
Variations of the Computation Algorithm for Subtraction	• Decomposing Where Needed First – The students will look for and decompose units prior to subtracting. The decomposition of these units can be recorded using place value drawings and/or within the problem. This can help students to avoid making common errors such as subtracting a smaller digit on top from a larger digit. Decomposing can be done in either direction (shown below from left to right).

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Tools Used for Adding and Subtracting

These tools and representations may be helpful as students are learning different strategies for addition and subtraction. They may also help students to see the different strategies.

		K.OA.1, K.OA.2, K.OA.3, K.OA.4, K.OA.5, I.OA.1, I.OA.2, I.OA.3, I.OA.4, I.OA.5 I.OA.6, I.OA.8, 2.OA.4
Five frames/ten frames

The student builds both addends to find the total or the student builds the total and removes the counters to subtract. | **5- and 10-frames**

Children can place small objects into 10-frames to show the ten as two rows of five and the extra ones within the next 10-frame, or work with strips that show ten ones in a column. | **Student:**
I used ten frames. I put 24 chips on 3 ten frames. Then, I counted out 8 more chips. 6 of them filled up the third ten frame. That meant I had 2 left over. 3 tens and 2 left over. That's 32. So, there are 32 apples on the table.

 |
| | **Student:**
I used ten frames. I had 6 ten frames- that's 60. I removed three ten frames because 30 students left the gym. There are 30 students left in the gym.

60 – 30 = 30 | |

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		K.OA.3, K.OA.4, K.OA.5, 1.OA.6, 1.OA.8, 2.OA.2, 3.OA.9, 4.NF.3b

Number bond diagram

The student illustrates how he/she decomposed or composed a given number.

Number-bond diagram and equation

17

10   7

$17 = 10 + 7$

Decompositions of teen numbers can be recorded with diagrams or equations.

total

7

4   3

		K.OA.1, K.OA.2, K.OA.3, K.OA.4, K.OA.5, 1.OA.1, 1.OA.2, 1.OA.3, 1.OA.4, 1.OA.5, 1.OA.6, 1.OA.8, 2.OA.1, 2.OA.2, 2.OA.4, 2.NBT.5, 2.NBT.6, 2.NBT.7, 2.NBT.8, 2.NBT.9, 2.MD.5, 2.MD.6, 2.MD.8, 3.OA.9, 3.NBT.2, 3.MD.1, 4.OA.3, 4.NF.3a, 4.NF.3c, 5.NF.2, 6.NS.3

Open number line

The student uses an open number line to show how they added, found a missing addend, or counted back. Students can represent different strategies on the open number line and can make "jumps" that make sense to him/her.

Student: I used a number line. First I jumped to 4, and then I jumped 5 more. Then, I jumped 5 more and got 15. Mrs. Smith has 15 cookies.

$4 + 5 + 6 = $

4   5   1   5

4   9   10   15

Student: I started at 354 and jumped 200. I landed on 554. I then made 8 jumps of 10 and landed on 634. I then jumped 6 to land on 640. Then I jumped 1 more and landed on 641. $354 + 287 = 641$

$354 + 287 = $

200

10 10 10 10 10 10 6 1

354   554 564 574 584 594 604 614 624 634 **640 641**

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K.OA.1, K.OA.2, 1.OA.1, 1.OA.2, 2.OA.1, 2.MD.5, 2.MD.8, 3.OA.8, 3.MD.1, 3.MD.2, 4.OA.3, 4.NF.3d, 5.NF.2
Tape diagram The student draws a picture that looks like a segment of tape. The number relationships between the quantities are indicated by the lengths of the bars in order to help the students make sense of the problem situation.	There are 25 students in a class. 15 of the students are girls. How many students are boys? If there are 8 trucks and 5 drivers, how many trucks do not have a driver? ...nd 3 green apples. How many total apples were there?

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Hundreds Chart	**Student:** I used a hundreds chart. I started at 63 and jumped down one row to 73. That means I moved 10 spaces. Then, I jumped down one more row (that's another 10 spaces) and landed on 83. So, there are 83 apples in the basket.	I.NBT.4, I.NBT.5, I.NBT.6, 2.NBT.5, 2.NBT.6
The student uses the hundreds chart to add or subtract. The student can make jumps of tens and ones to find their total.	$$63 + 10 = 73$$ $$73 + 10 = 83$$ **Student:** I used a hundreds chart. I started at 49 and jumped down one row to 59. That means I moved 10 spaces. Next, I jumped down one more row (that's another 10 spaces) and landed on 69. Then I moved forward three spaces to 72. $$49 + 23 =$$ $$49 + 10 + 10 + 3 = 72$$	
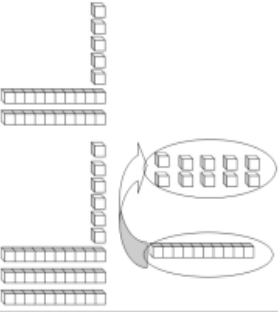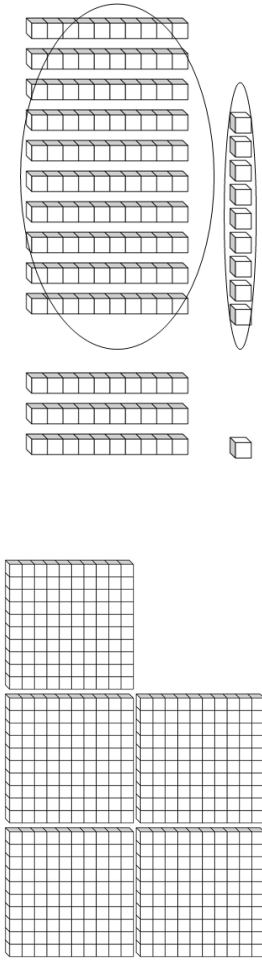
Place Value Blocks	I.NBT.4, I.NBT.5, I.NBT.6, 2.NBT.5, 2.NBT.6, 2.NBT.7, 2.NBT.8, 2.NBT.9, 2.MD.8, 3.NBT.2, 4.NBT.5, 5.NBT.7
The student uses base ten blocks to add or subtract numbers. A variety of strategies can be used when adding using this tool.	**Student:** I used place value blocks and made a pile of 36 and a pile of 25. Altogether, I had 5 tens and 11 ones. 11 ones is the same as one ten and one one left over. So, I really had 6 tens and 1 one. That makes 61.
	Student: I used place value blocks. I made a pile of 354. I then added 287. That gave me 5 hundreds, 13 tens and 11 ones. I noticed that I could trade some pieces. I had 11 ones, and traded 10 ones for a ten. I then had 14 tens, so I traded 10 tens for a hundred. I ended up with 6 hundreds, 4 tens and 1 one. So, 354 + 287 = 641

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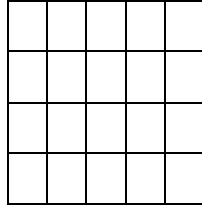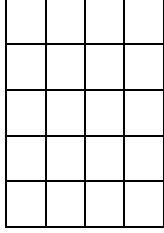
Strategy	Explanation	Examples
Equal groups		
Array/area model		
Repeated addition or skip counting		
Partial products		
Breaking factors into smaller factors		

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Making a landmark or friendly number	
Doubling and halving	
Repeated subtraction or sharing/dealing out	
Multiplying up	
Partial quotients	
Proportional reasoning	

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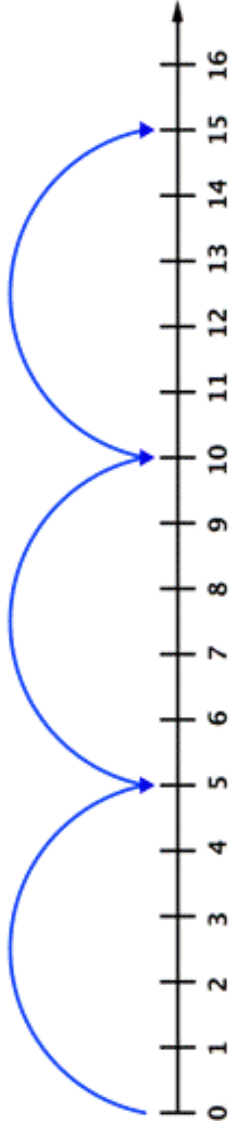
Strategy	Explanation	Examples	Supporting Standards
Equal Groups	The student represents a problem situation or numerical expression using objects, a drawing or fingers.	This task can also be solved by drawing pictures of equal groups. 4 groups of 6 equals 24 objects. A student can also reason through the problem mentally or verbally, "I know 6 and 6 are 12. 12 and 12 are 24. Therefore, there are 4 groups of 6 giving a total of 24 desks in the classroom."	3.OA.3, 3.OA.4, 3.OA.5, 4.OA.2
Array	The student thinks of one way to configure their rows and columns and constructs an array. As rows and columns can be used interchangeably.	There are 20 desks in the classroom. If the teacher puts 5 desks in each row, how many rows are there? This task can be solved by drawing an array by putting 5 desks in each row. This is an array model. 4 x 5 or 5 x 4	3.OA.5

Strategies for Multiplication

	4.NBT.5, 5.NBT.5

Area Model	Students develop flexibility in breaking numbers apart and develop understanding of general base-ten multiplication methods.	

$36 \times 94 = (30 + 6) \times (90 + 4)$
$= (30 + 6) \times 90 + (30 + 6) \times 4$
$= 30 \times 90 + 6 \times 90 + 30 \times 4 + 6 \times 4.$

Computation of 36×94 connected with an area model

	90	+ 4
30	$30 \times 90 =$ 3 tens × 9 tens = 27 hundreds = 2700	$30 \times 4 =$ 3 tens × 4 = 12 tens = 120
+ 6	$6 \times 90 =$ 6 × 9 tens = 54 tens = 540	$6 \times 4 = 24$

The products of like base-ten units are shown as parts of a rectangular region.

Computation of 36×94: Ways to record general methods

Showing the partial products

```
    94
  × 36
 ─────
    24   thinking:  6 × 4
   540              6 × 9 tens
   120              3 tens × 4
  2700              3 tens × 9 tens
 ─────
  3384
```

Recording the carries below for correct place value placement

```
      94
    × 36
  ─────
     44
    720
  ─────
   3384
```

because we are multiplying by 3 tens in this row

These proceed from right to left, but could go left to right. On the right, digits that represent newly composed tens and hundreds are written below the line instead of above 94. The digits 2 and 1 are surrounded by a blue box. The 1 from $30 \times 4 = 120$ is placed correctly in the hundreds place and the digit 2 from $30 \times 90 = 2700$ is placed correctly in the thousands place. If these digits had been placed above 94, they would be in incorrect places. Note that the 0 (surrounded by a yellow box) in the ones place of the second line of the method on the right is there because the whole line of digits is produced by multiplying by 30 (not 3).

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Decomposing a Factor	Students can easily break apart factors to make simpler problems or multiplication facts they are familiar with.	Students are introduced to the distributive property of multiplication over addition as a strategy for using products they know to solve products they don't know. Students would be using mental math to determine a product. Here are ways that students could use the distributive property to determine the product of 7 x 6. Again, students should use the distributive property, but can refer to this in informal language such as "breaking numbers apart".	3.OA.5, 4.NBT.5, 5.NBT.5

Student 1
7 x 6
7 x 5 = 35
7 x 1 = 7
35 + 7 = 42

Student 2
7 x 6
7 x 3 = 21
7 x 3 = 21
21 + 21 = 42

Student 3
7 x 6
5 x 6 = 30
2 x 6 = 12
30 + 12 = 42

Another example if the distributive property helps students determine the products and factors of problems by breaking numbers apart. For example, for the problem 7 x 8 = ?, students can decompose the 7 into a 5 and 2, and reach the answer by multiplying 5 x 8 = 40 and 2 x 8 =16 and adding the two products (40 +16 = 56).

5 x 8

2 x 8

			5.NBT.5
Doubling and Halving	Students can manipulate factors using doubling and halving strategy to multiply "friendly numbers."	Example: There are 25 dozen cookies in the bakery. What is the total number of cookies at the bakery? Student A Response: 25 x 12 I doubled 25 and cut 12 in half to get 50 x 6. 50 x 6 = 300	
Standard Algorithm	Students multiply multi-digit numbers fluently using the standard algorithm.	This standard refers to fluency which means accuracy (correct answer), efficiency (a reasonable amount of steps), and flexibility (using strategies such as the distributive property or breaking numbers apart also using strategies according to the numbers in the problem.) Multiply the ones first. / Then multiply the tens and place the result underneath. Remember, the 3 in 34 is signifying 30. / Then add. $\begin{array}{r} 34 \\ \times\ 5 \\ \hline 20 \end{array}$ $5 \times 4 = 20$ $\begin{array}{r} 34 \\ \times\ 5 \\ \hline 20 \\ 150 \end{array}$ $5 \times 30 = 150$ $\begin{array}{r} 34 \\ \times\ 5 \\ \hline 20 \\ +\ 150 \\ \hline 170 \end{array}$	

Supporting the Common Core State Standards for Mathematics

Strategies for Multiplication

Tools Used for Multiplication

These tools and representations may be helpful as students are learning different strategies for multiplication. They may also help students to see the different strategies as they are recorded using the tools.

Number bond diagram The student illustrates how he/she decomposed or composed a given number.	Student: The number bond displays the fact family of 9 and 5. $9 \times 5 = 45$ $5 \times 9 = 45$ $45 \div 9 = 5$ $45 \div 5 = 9$
Open number line The student uses an open number line to show how they multiplied. Students can represent equal "jumps."	A number line could also be used to show equal jumps. For example $3 \times 5 = 15$ (3 jumps of 5)

Tape diagram The student draws a picture that looks like a segment of tape. The number relationships and the unknown are indicated.	**A two-step problem with diagram showing problem situation and equations showing the two parts** Carla has 4 packages of silly bands. Each package has 8 silly bands in it. Agustin is supposed to get 15 fewer silly bands than Carla. How many silly bands should Agustin get? Carla: [8] [8] [8] [8] Agustin: [ ] (15) C = number of Carla's silly bands A = number of Agustin's silly bands $C = 4 \times 8 = 32$ $A + 15 = C$ $A + 15 = 32$ $A = 17$ *Students may be able to solve this problem without writing such equations.*
Hundreds Chart The student uses the hundreds chart to multiply. Starting at 0 and making equal jumps of a number.	**Student:** I used a hundreds chart. I started at 0 and made 6 jumps of 4 and arrived at 24. $6 \times 4 = 24$ | 100									
---	---	---	---	---	---	---	---	---	---
90	91	92	93	94	95	96	97	98	99
80	81	82	83	84	85	86	87	88	89
70	71	72	73	74	75	76	77	78	79
60	61	62	63	64	65	66	67	68	69
50	51	52	53	54	55	56	57	58	59
40	41	42	43	44	45	46	47	48	49
30	31	32	33	34	35	36	37	38	39
20	21	22	23	24	25	26	27	28	29
10	11	12	13	14	15	16	17	18	19
0	1	2	3	4	5	6	7	8	9	

Table 1. Common addition and subtraction situations.[3]

	Result Unknown	Change Unknown	Start Unknown
Add to	Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now? $2 + 3 = ?$	Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were five bunnies. How many bunnies hopped over to the first two? $2 + ? = 5$	Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were five bunnies. How many bunnies were on the grass before? $? + 3 = 5$
Take from	Five apples were on the table. I ate two apples. How many apples are on the table now? $5 – 2 = ?$	Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat? $5 – ? = 3$	Some apples were on the table. I ate two apples. Then there were three apples. How many apples were on the table before? $? – 2 = 3$

	Total Unknown	Addend Unknown	Both Addends Unknown[4]
Put Together/ Take Apart[5]	Three red apples and two green apples are on the table. How many apples are on the table? $3 + 2 = ?$	Five apples are on the table. Three are red and the rest are green. How many apples are green? $3 + ? = 5, 5 – 3 = ?$	Grandma has five flowers. How many can she put in her red vase and how many in her blue vase? $5 = 0 + 5, 5 = 5 + 0$ $5 = 1 + 4, 5 = 4 + 1$ $5 = 2 + 3, 5 = 3 + 2$

	Difference Unknown	Bigger Unknown	Smaller Unknown
Compare[6]	(“How many more?” version): Lucy has two apples. Julie has five apples. How many more apples does Julie have than Lucy? (“How many fewer?” version): Lucy has two apples. Julie has five apples. How many fewer apples does Lucy have than Julie? $2 + ? = 5, 5 – 2 = ?$	(Version with “more”): Julie has three more apples than Lucy. Lucy has two apples. How many apples does Julie have? (Version with “fewer”): Lucy has 3 fewer apples than Julie. Lucy has two apples. How many apples does Julie have? $2 + 3 = ?, 3 + 2 = ?$	(Version with “more”): Julie has three more apples than Lucy. Julie has five apples. How many apples does Lucy have? (Version with “fewer”): Lucy has 3 fewer apples than Julie. Julie has five apples. How many apples does Lucy have? $5 – 3 = ?, ? + 3 = 5$

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3. Adapted from Boxes 2–4 of *Mathematics Learning in Early Childhood,* National Research Council (2009, pp. 32–33).

4. These take apart situations can be used to show all the decompositions of a given number. The associated equations, which have the total on the left of the equal sign, help children understand that the = sign does not always mean *makes or results in* but always does mean *is the same number as.*

5. Either addend can be unknown, so there are three variations of these problem situations. "Both Addends Unknown" is a productive extension of this basic situation, especially for small numbers less than or equal to 10.

6. For the Bigger Unknown or Smaller Unknown situations, one version directs the correct operation (the version using *more* for the bigger unknown and using *less* for the smaller unknown). The other versions are more difficult.

Table 2. Common multiplication and division situations.[7]

	Unknown Product 3 × 6 = ?	Group Size Unknown ("How many in each group?" Division) 3 × ? = 18 and 18 ÷ 3 = ?	Number of Groups Unknown ("How many groups?" Division) ? × 6 = 18 and 18 ÷ 6 = ?
Equal Groups	There are 3 bags with 6 plums in each bag. How many plums are there in all? *Measurement example.* You need 3 lengths of string, each 6 inches long. How much string will you need altogether?	If 18 plums are shared equally into 3 bags, then how many plums will be in each bag? *Measurement example.* You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be?	If 18 plums are to be packed 6 to a bag, then how many bags are needed? *Measurement example.* You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have?
Arrays,[8] Area[9]	There are 3 rows of apples with 6 apples in ea ch row. How many apples are there? *Area example.* What is the area of a 3 cm by 6 cm rectangle?	If 18 apples are arranged into 3 equal rows, how many apples will be in each row? *Area example.* A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it?	If 18 apples are arranged into equal rows of 6 apples, how many rows will there be? *Area example.* A rectangle has area 18 square centimeters. If one side is 6 cm long, how long is a side next to it?
Compare	A blue hat costs $6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost? *Measurement example.* A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be 3 times as long?	A red hat costs $18 and that is 3 times as much as a blue hat costs. How much does a blue hat cost? *Measurement example.* A rubber band is stretched to be 18 cm long and that is 3 times as long as it was at first. How long was the rubber band at first?	A red hat costs $18 and a blue hat costs $6. How many times as much does the red hat cost as the blue hat? *Measurement example.* A rubber band was 6 cm long at first. Now it is stretched to be 18 cm long. How many times as long is the rubber band now as it was at first?
General	$a \times b = ?$	$a \times ? = p$ and $p \div a = ?$	$? \times b = p$ and $p \div b = ?$

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7. The first examples in each cell are examples of discrete things. These are easier for students and should be given before the measurement examples.
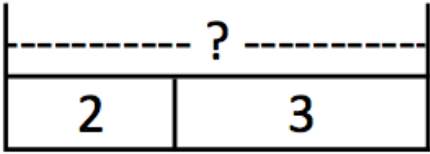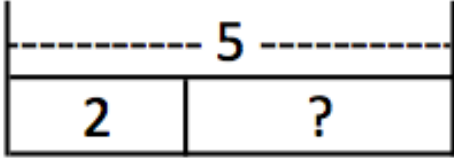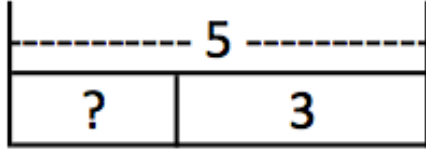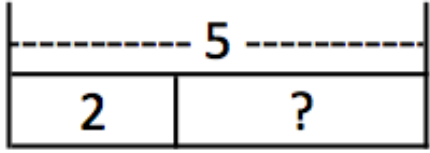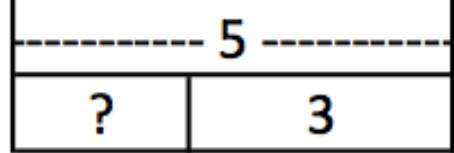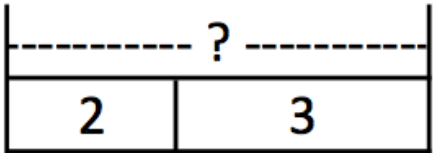
8. The language in the array examples shows the easiest form of array problems. A harder form is to use the terms rows and columns: The apples in the grocery window are in 3 rows and 6 columns. How many apples are in there? Both forms are valuable.

9. Area involves arrays of squares that have been pushed together so that there are no gaps or overlaps, so array problems include these especially important measurement situations.
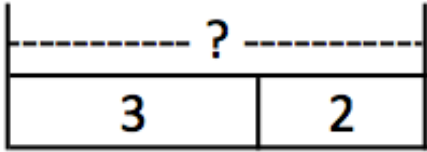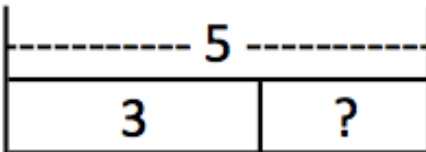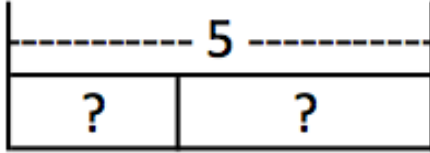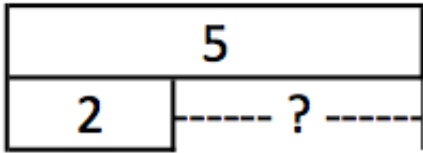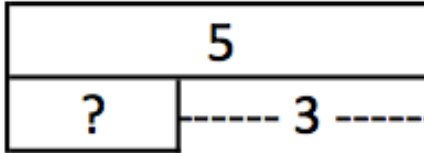
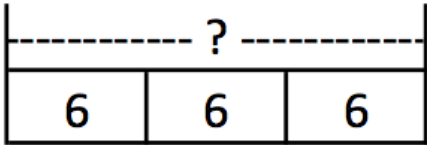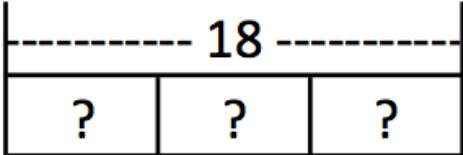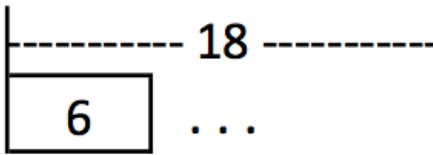# Tape Diagrams
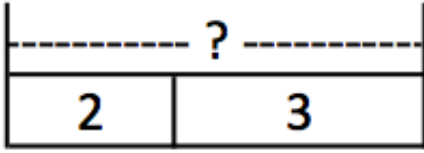
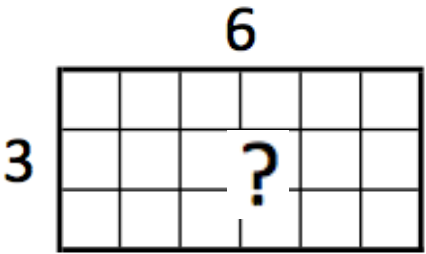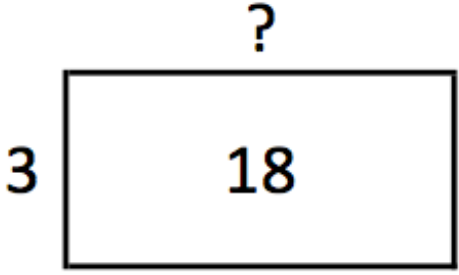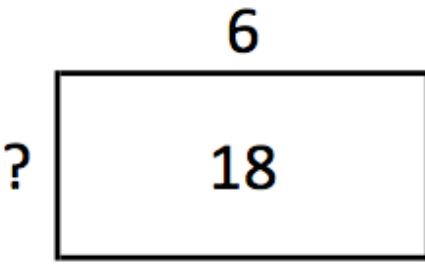|	
When?	To support students as a tool for representing and solving word problems.
Why?	Students often struggle with word problems, unsure of the relationships they describe and what they are asked to find. Drawing tape diagrams allows students to make sense of the problem situation by creating visual representations of the quantities involved and their relationships. This process delays answer getting and enables students to understand the context of the problem. Students practice decontextualizing the situation by taking it from words to the representation and then they re-contextualize the situation when they interpret their solution.
What?	A drawing that looks like a segment of tape, used to illustrate number relationships. Also known as a strip diagram, bar model, fraction strip, or length model. Tape diagrams help students move from concrete to pictorial to abstract understanding.
	Supports solving word problems involving:
	- Addition - Subtraction - Multiplication - Division - Comparisons - Fractions - Algebraic equations - Ratios and Proportional Relationships
What it looks like?	Tape diagrams will need to be taught to students and practiced with word problems throughout the year so that students feel comfortable using the tool for solving word problems. - Multiple reads each with a purpose to fully understand the problem, retell and discuss as needed - Students write a sentence that will contain the answer - Students draw a model, filling in and labeling as much information as possible - Solve and check

Adapted from <u>Bar Modeling: A Problem-solving Tool</u> By Yeap, Ban Har, PhD

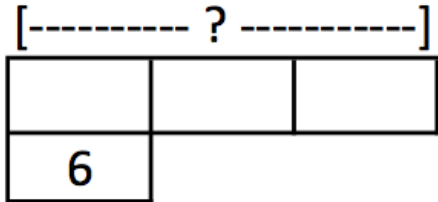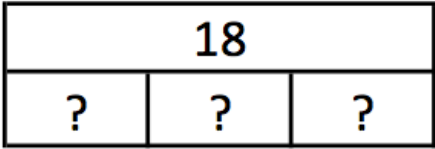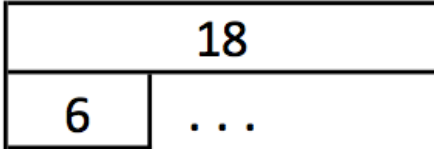|	RESULT UNKNOWN	CHANGE UNKNOWN	START UNKNOWN
ADD TO	<u>Two</u> bunnies sat on the grass. <u>Three</u> **more** bunnies hopped there. How many bunnies are on the grass now? ┌─────── ? ───────┐ │ **2** │ **3** │ 2 + 3 = ?	<u>Two</u> bunnies were sitting on the grass. **Some more** bunnies hopped there. Then there were <u>five</u> bunnies. How many bunnies hopped over to the first two? ┌─────── 5 ───────┐ │ **2** │ **?** │ 2 + ? = 5	**Some** bunnies were sitting on the grass. <u>Three more</u> bunnies hopped there. Then there were <u>five</u> bunnies. How many bunnies were on the grass before? ┌─────── 5 ───────┐ │ **?** │ **3** │ ? + 3 = 5
TAKE FROM	<u>Five</u> apples were on the table. I <u>ate two</u> apples. How many apples are on the table now? ┌─────── 5 ───────┐ │ **2** │ **?** │ 5 – 2 = ?	<u>Five</u> apples were on the table. I **ate some** apples. Then there were <u>three</u> apples. How many apples did I eat? ┌─────── 5 ───────┐ │ **?** │ **3** │ 5 – ? = 3	**Some** apples were on the table. I <u>ate two</u> apples. Then there were <u>three</u> apples. How many apples were on the table before? ┌─────── ? ───────┐ │ **2** │ **3** │ ? – 2 = 3

	TOTAL UNKNOWN	**ADDEND UNKNOWN**	**BOTH ADDENDS UNKNOWN**
PUT TOGETHER / TAKE APART	Three red apples and two green apples are on the table. How many apples are on the table? $3 + 2 = ?$	Five apples are on the table. Three are red and **the rest** are green. How many apples are green? $3 + ? = 5$ $5 - 3 = ?$	Grandma has five flowers. How many can she put in her red vase and how many in her blue vase? $5 = 0 + 5, \quad 5 = 5 + 0$ $5 = 1 + 4, \quad 5 = 4 + 1$ $5 = 2 + 3, \quad 5 = 3 + 2$

	DIFFERENCE UNKNOWN	**BIGGER UNKNOWN**	**SMALLER UNKNOWN**
COMPARE	Lucy has two apples. Julie has five apples. **How many more** apples does Julie have than Lucy? Lucy has two apples. Julie has five apples. **How many fewer** apples does Lucy have than Julie? $2 + ? = 5, \ 5 - 2 = ?$	Julie has **three more** apples than Lucy. Lucy has two apples. How many apples does Julie have? Lucy has **three fewer** apples than Julie. Lucy has two apples. How many apples does Julie have? $2 + 3 = ?, \ 3 + 2 = ?$	Julie has **three more** apples than Lucy. Julie has five apples. How many apples does Lucy have? Lucy has **three fewer** apples than Julie. Julie has five apples. How many apples does Lucy have? $5 - 3 = ?, \ ? + 3 = 5$

Math in Common: Strategies for Implementation

	UNKNOWN PRODUCT	GROUP SIZE UNKNOWN	NUMBER OF GROUPS UNKNOWN
EQUAL GROUPS	There are 3 bags with 6 plums **in each** bag. How many plums are there **in all**?	If 18 plums are shared **equally into 3** bags, then how many plums will be in each bag?	If 18 plums are to be packed **6 to a bag**, then **how many bags** are needed?
	You need 3 lengths of string, **each 6 inches long**. How much string will you need **altogether**?	You have 18 inches of strin, which you will cut into **3 equal pieces**. How long will **each** piece of string be?	You have 18 inches of string. which you will cut into pieces that are 6 inches long. **How many pieces** of string will you have?
| | ```
|-------- ? --------|
|  6  |  6  |  6  |
``` | ```
|-------- 18 --------|
|  ?  |  ?  |  ?  |
``` | ```
|-------- 18 ----------
|  6  |   . . .
``` |
| | $3 \times 6 = ?$ | $3 \times ? = 18$
$18 \div 3 = ?$ | $? \times 6 = 18$
$18 \div 6 = ?$ |

	UNKNOWN PRODUCT	GROUP SIZE UNKNOWN	NUMBER OF GROUPS UNKNOWN
ARRAYS, AREA	There are 3 rows of apples with 6 apples in **each row**. How many apples are there?	If 18 apples are arranged into 3 equal rows, how many apples will be in **each row?**	If 18 apples are arranged into equal rows of 6 apples, **how many rows** will there be?
	What is the area of a 3cm by 6cm rectangle?	A rectangle has area 18 square centimeters. If one side is 3 cm long, **how long is a side** next to it?	A rectangle has area 18 square cm. If one side is 6cm long, **how long is a side** next to it?
	6 / 3 / ?	? / 3 / 18	6 / ? / 18
	$3 \times 6 = ?$	$3 \times ? = 18$ $18 \div 3 = ?$	$? \times 6 = 18$ $18 \div 6 = ?$

	UNKNOWN PRODUCT	**GROUP SIZE UNKNOWN**	**NUMBER OF GROUPS UNKNOWN**
COMPARE	A blue hat costs $6. A red hat costs **3 times as much** as the blue hat. How much does the red hat cost?	A red hat costs **$18** and that **is 3 times as much** as a blue hat costs. How much does a blue hat cost?	A red hat costs $18 and a blue hat costs $6. **How many times as much** does the red hat cost as the blue hat?
	A rubber band is 6cm long. How long will the rubber band be when it is stretched to be **3 times as long**?	A rubber band is stretched to be 18 cm long and that is **3 times as long** as it was at first. **How long was** the rubber band **at first?**	A rubber band was 6 cm long at first. How it is stretched to be 18 cm long. **How many times as long** is the rubber band now as it was at first?
	[---------- ? ----------] 6 $$3 \times 6 = ?$$	18 ? ? ? $$3 \times ? = 18$$ $$18 \div 3 = ?$$	18 6 . . . $$? \times 6 = 18$$ $$18 \div 6 = ?$$

Addition and Subtraction:

Putting Quantities next to one another suggests an additive relationship.

[Quantity 1] + [Quantity 2] = [Total Quantity]

Total is indicated by length.

Quantities can be compared by juxtaposition.

Part - Whole:

Find the whole given two parts.

1: Annie has 37 cards. Bonnie has 29 cards. How many cards do they have altogether?

Find a part given the whole and the other part.

2: There are 295 children at a school. 127 stayed for extracurricular activities after school. How many did not stay for an activity?

Comparison:

3: Alan bought a bag with 29 jelly beans. He gave 13 to his little brother. How many jelly beans did Alan keep for himself?

4: Carl has 13 games in his collection. Daniel has 7 more than Carl. How much do they have altogether.*

Additional Problems:

5: Mary made 686 biscuits. She sold some of them. If 298 were left over, how many biscuits did she sell?

6: Joe saved $184. He saved $63 more than Trevor. How much did Trevor save?

Multiplication and Division:

Consider that we now are counting groups in addition to objects within a group.

 5 x n

Find the whole given the number of parts and the size of one part.

7: Seven friends went out for a nice dinner. They decided to split the bill evenly amongst themselves and paid $23 each. What was the total money left by the seven friends?

Find the size of each part given the whole and the number of parts.

8: The grand prize for a contest was $270. Five winners came forward to claim the prize. Their claims were legitimate and it was decided to split the prize amongst the five winners. How much did each winner take home?

Find the number of parts given the whole and the size of 1 part.

9: A box containing 96 pencils is opened and shared equally among some children. Each child receives 8 pencils. How many children shared the box of pencils?

Multiplicative Comparison:

Given the smaller quantity and the multiple, find the larger quantity.

10: Eric has saved $13.25. His sister, Fiona, has saved 3 times as much. How much has Fiona saved?

Given the larger quantity and the multiple, find the smaller quantity.

11: Gabriel harvested 124 lemons from the large tree in his yard. This was four times as much as he harvested from the small tree in his yard. How many did he collect from the small tree?

Given the two quantities, find the multiple (scale factor).

12: Hermione was able to catch 90 fireflies. Ingrid was able to catch 18 fireflies. How many times as many fireflies did Hermione catch compared to Ingrid?

Fractions:

Given the whole and the fraction, determine the quantity the fraction represents.

13: Jerry bought a large gummi bear weighing 64 ounces. He and his friends ate 3/4 of it. How many ounces did they eat?

Given the quantity represented by the fraction and the fraction, determine the quantity representing the whole.

14: Kendra bought a batch of cookies, 2/3 of which were chocolate chip. If 30 of them were chocolate chip, how many total cookies did Kendra buy?

Comparisons:

15: Lily spent 2/5 of her cash on a CD which cost $15. How much money did Lily have to begin with?

16: Maya has a garden. 1/4 of the garden is seeded with tomato plants. 1/3 of the garden is peppers. The rest of the garden will be for flowers. What fraction of the garden will be flowers?

17: Nena had 180 peaches to sell. She sold 1/3 of them on Friday and 1/2 on Saturday. How many peaches will she still have going in to Sunday?

18: If Oscar spent 5/6 of his money and only has $12 left, how much did he have to begin with?

19: Paul bought 280 blue and red cups for a party. He used 1/3 of the blue cups and 1/2 of the red ones. If he had an equal number of blue and red cups left over, how many did he use altogether?

Ratios and Proportional Relationships:

20. There are several pieces of fruit in a box. Twenty-four are limes, the rest are lemons. The ratio of limes to lemons is 4 to 1. Find the number of lemons.

21. Quincy is mixing orange paint and needs to use 3 parts of red to every 5 parts of yellow. If he is buying 24 cans of red paint, how many cans should he buy of the yellow paint?

22.* Slimy Gloopy mixture is made by mixing glue and liquid laundry starch in a ratio of 3 to 2. How much glue and how much starch is needed to make 85 cups of Slimy Gloopy mixture?

23.* Yellow and blue paint were mixed in a ratio of 5 to 3 to make green paint. After 14 liters of blue paint were added, the amount of yellow and blue paint in the mixture was equal. How much green paint was in the mixture at first?

24. Jim and Jesse each had the same amount of money. Jim spent $58 to fill the car up with gas for a road-trip. Jesse spent $37 buying snacks for the trip. Afterward, the ratio of Jim's money to Jesse's money is 1:4. How much money did each have at first?

Other Problems:

A. Ali has $8 more than Sid. Trina has $6 less than Ali. The three of them have $76 in all. Find the amount of money that each of them has.

B: 88 children attended swim camp. One-third of the boys and three-sevenths of girls wore goggles. If 34 students wore goggles, how many girls wore goggles? *

DUSD CCSS-Math
Lesson Study Process

Outcomes:
- Deeper understanding of the Standards for Mathematical Practice
- Deeper understanding of new CCSS-M standards for your grade level
- Deeper understanding of the Launch, Explore, Summarize instructional model
- A polished lesson for future use
- New knowledge that can be applied to future lessons and math content
- Opportunity to collaborate with colleagues? Priceless!

Lesson Study Design Process
1. Choose a lesson from an upcoming unit that your grade level team would like to explore and build a deeper understanding.
2. Use the Launch, Explore, Summarize instructional model as a guide, and select a segment of the model to strengthen as a team.
3. Decide on a Standard for Mathematical Practice to emphasize in your lesson.
4. Include an engagement structure that you want to explore and may support your lesson goals.
5. Look for ways to include student writing in the launch, explore, and/or summarize portion of the lesson.

Lesson Study Sequence:
1. Two consecutive Grade Level PLC sessions to design the lesson around the attached components
2. One 45-60 minute math lesson taught by one team member and observed by rest of team
3. 15 minute break
4. One 60-90 minute Debrief session with grade level team
5. Lunch depending on school site schedule
6. One 45-60 minute math lesson taught by another team member to another class with revisions from the debrief
7. One 60-90 minute Debrief session with grade level team
8. Repeat cycle if you desire and your debrief session times allow

Suggested Questions to Guide Lesson Planning
1. What are a few things you want to accomplish in this lesson for your students?
2. What might be some of the student outcomes you would anticipate?
3. What might be some evidence to collect which would support your student outcomes?
4. How might you know when your students have been successful in the lesson?
5. What would the students be doing to show their understanding of the new content?
6. What would the students be doing to show their increasing skill with the selected Standard for Mathematical Practice?
7. What might be some teaching strategies to ensure student understanding?
8. As you envision your lesson, what possible misconceptions or student responses might you plan for?
9. What are some possible difficulties students may encounter, and how might you address them?
10. How might this formative feedback inform your next steps with your class?
11. What might you want to be sure and do well during the lesson?

Lesson Study Protocol

The following protocol guidelines are meant to facilitate the lesson observation and debriefing process. Although these guidelines are meant to make these activities more constructive and efficiently organized, they are not meant to minimize the critical or reflective nature of the feedback session.

Observing the lesson:

1. The observers, including the teachers who helped plan the lesson, should NOT interfere with the natural process of the lesson (e.g., by helping students with a problem). However, observers are permitted to circulate around the classroom during seatwork, as well as communicate with students for clarifying purposes only (e.g., if they could not clearly hear what a student was saying). Otherwise, observers should be seated at the back and sides of the classroom.
2. It is a good idea for observers to note their observations on the lesson plan itself. This procedure will not only help observers focus on the goals and activities of the lesson, but also help them organize their feedback for later.
3. It is also a good idea for observers to distribute observations among themselves. For example, a few clusters of observers could watch assigned groups of students, another observer (usually one of the planning teachers) could keep time, etc. The teacher should also prepare for this observation by distributing seating charts among the observers (if seating charts are not available, s/he could place nametags on each student), so that observers can conveniently refer to the children by name when discussing their observations and sharing their feedback.

Preparing for the feedback session:

1. Instead of discussing the lesson immediately after it has been taught, the entire group should take a break to relax and gather their thoughts.
2. The group who planned the lesson should assign roles among themselves in order to help keep the discussion focused and on track. These roles include: moderator/ facilitator (usually a member of the planning group besides the teacher who taught the lesson), timekeeper, and recorder(s).
3. The teachers who planned the lesson should sit together around a table during the debrief session. The purpose of this setup is to emphasize the idea that the entire group (not just the teacher who taught the lesson) is receiving the feedback.

Suggestions for sharing feedback about the lessons:

1. The facilitator selected by the grade level team should begin the debrief session by (1) outlining the agenda for the discussion (e.g., "first we will hear from the teachers who planned the lesson, and then…"); and by (2) briefly introducing the goals of the planning group.

2. The teacher who taught the lesson should have the first opportunity to comment on his/ her reactions to the lesson, followed by the other planning group members. S/he should address what actually occurred during the lesson (e.g., what worked, what did not work, what could be changed about the lesson, etc.).

3. The planning teachers should also raise questions/ issues that were raised during the planning sessions, and describe how these concerns were addressed by the instructional decisions the team made for the study lesson. If the debrief session is after the second implementation of a study lesson, the planning members should clarify what changes were made between the two lessons, and how these changes related to the goals of the lesson.

4. The planning teachers should direct the observers to give them feedback that is related to the goals of the lesson. The observers can then share feedback about the lesson that helps the planning teachers address these goals. For example, observers could share their suggestions about how they might have done something differently in their own classes. Or, they could ask the planning teachers about their rationales for making certain decisions about the lesson (e.g., "Why did you choose those numbers for that problem?").

5. When observers share their feedback, they should begin on a positive note by thanking the teacher who taught the lesson and discussing what they liked about the lesson. Observers should then share critical feedback by supporting their statements with concrete evidence. For example, they could comment on specific observations from this particular lesson (e.g., "I saw student X do this…"), or make suggestions that draw upon their own experiences (e.g., "When I taught a similar lesson, I did (blank) differently because…").

6. Each observer should comment on a specific aspect of the lesson, and then give other observers the opportunity to comment on this point or related aspects of the lesson. This procedure prevents the feedback session from becoming dominated by one observer, and allows others to share their insights. If an observer would like to share something that is not being discussed at that point, s/he can write it down for later.

7. Similarly, the teacher(s) who planned/ taught the lesson should wait until a few comments about a particular aspect of a lesson have been received before responding to the observers. This waiting etiquette prevents the discussion from becoming a point-volleying session, and allows all participants to voice and absorb the feedback in a reflective manner. In addition, the facilitator should be responsible for proactively keeping the debriefing session on track.

8. The timekeeper should remind the group when time is running short, so that the group can meaningfully wrap up their debriefing session. If an outside advisor is present, the feedback session should end with general comments from that person.

Suggested Questions to Guide Lesson Debriefs and Reflection

1. What might be some of your impressions of the math lesson and the use of the selected Standard for Mathematical Practice?
2. What might be some of your thoughts about the lesson?
3. How do you feel the students did with the content and/or the Standard for Mathematical Practice?
4. What might be some things you noticed about the student understanding of the new content?
5. What might be your observations of their use of the Standard for Mathematical Practice?
6. What did you notice about the three phases of the lesson?
7. What are some things you did that contributed to student success with the new math content?
8. So what kind of things were you doing to emphasize the Standard for Mathematical Practice?
9. What kinds of instructional approaches might you have been using to help students understand the content?
10. What learning will you take away for future math lessons?
11. What might be your next step as a PLC and in your own classroom?
12. If this were your classroom, what would be your next steps based on the student understanding and misunderstandings that were present?
13. How might you apply what you learned in this lesson to other lessons?